Dynamics on the Riemann Sphere

A Bodil Branner Festschrift

Poul G. Hjorth
Carsten Lunde Petersen

Editors

European Mathematical Society

Editors:

Associate Professor
Carsten Lunde Petersen
IMFUFA
Roskilde Universitetscenter
Universitetsvej 1, Postbox 260
DK-4000 ROSKILDE
DENMARK

Associate Professor
Poul G. Hjorth
Department of Mathematics
Technical University of Denmark
Building 303, Matematiktorvet
DK-2800 Kgs. LYNGBY
DENMARK

2000 Mathematics Subject Classification 37F45

Bibliographic information published by Die Deutsche Bibliothek

Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie;
detailed bibliographic data are available in the Internet at http://dnb.ddb.de.

ISBN 3-03719-011-6

Contact address:

European Mathematical Society Publishing House
Seminar for Applied Mathematics
ETH-Zentrum FLI C4
CH-8092 Zürich
Switzerland

Phone: +41 (0)44 632 34 36
Email: info@ems-ph.org
Homepage: www.ems-ph.org

Cover illustration courtesy Mitsuhiro Shishikura, Kyoto
Printed in Germany

9 8 7 6 5 4 3 2 1

Contents

Introduction

The papers collected in this volume were all written in celebration of Bodil Branner's 60 year birthday. Most of them were presented at the 'Bodil Fest', a symposium on holomorphic dynamics held in June 2003 at the 'Søminestationen' in Holbæk, Denmark.

JOHN MILNOR gives an exhaustive survey of the so called Lattès maps, their history, their properties and significance within holomorphic dynamics in general and within Thurston theory in particular.

CARSTEN LUNDE PETERSEN and TAN LEI survey the Branner-Hubbard motion and illustrate its power by old and new examples.

MICHAEL LYUBICH and ARTHUR AVILA study the Hausdorff dimension of the Julia sets of the sequence of infinitely renormalizeable real quadratic polynomials with p-periodic combinatorics closest to the Chebychev polynomial, $p \to \infty$, using their Poincaré Series technicque.

ARNAUD CHERITAT surveys his joint work with XAVIER BUFF on quadratic Siegel disks with prescribed boundary regularity.

ROBERT DEVANEY, DANIEL M. LOOK, MONICA MORENO ROCHA, PRADIPTA SEAL, STEFAN SIEGMUND, and DAVID UMINSKY portray the family of quartic rational maps $z \mapsto z^2 + \lambda/z^2$ which exhibits many interesting properties, and they pose several questions about this family.

PASCALE ROESCH subsequently answers one of these questions affirmatively.

TOMOKI KAWAHIRA studies small perturbations of geometrically finite maps into other geometrically finite maps which are (semi)-conjugate on the Julia set to the original map.

WOLF JUNG presents his interesting thesis work on self-homeomorphisms of the Mandelbrot set. He shows among other things that the group of such homeomorphisms has the cardinality of $\mathbb{R}$.

NURIA FAGELLA and CHRISTIAN HENRIKSEN study the natural complexifications of the so-called standard maps and Arnold disks in the corresponding complexifications in parameterspace, of the irrational Arnold tongues for rotation numbers yielding Herman rings.

TAN LEI surveys and extends the results of the unpublished thesis of PIA WILLUMSEN, who was a Ph.D. student of Bodil Branner.

ADRIEN DOUADY, in the final paper, describes Branner-Hubbard motions of compact sets in the plane, poses several convergence conjectures and proves new results on semi-hyperbolic parameters in the Mandelbrot set.

Bodil Branner is a graduate of Aarhus University. In 1967 she submitted her master thesis in the area of algebraic topology. Since 1969 she has worked, first as an assistant

professor, later as associate professor of mathematics at the Department of Mathematics, Technical University of Denmark, which is situated in Lyngby, near Copenhagen.

In the summer of 1983 Branner had the fortune to be introduced to holomorphic dynamics when she met Adrien Douady and John H. Hubbard.

All three participated in the *Chaos* workshop at the Niels Bohr Institute at the University of Copenhagen, organized by Predrag Cvitanovic. The workshop poster displayed the Douady Rabbit. Branner had learned from Cvitanovic that Hubbard had generalized kneading sequences – used to classify one-dimensional real uni-modal maps such as real quadratic polynomials – to the complex setting, the Hubbard trees. At the time Branner worked on iterations of real cubic polynomials. This was initiated by Peter Leth Christiansen who in 1980 suggested, as a topic for a student's master thesis, the Nature paper of 1976 by Robert May on *Complicated Behavior of Simple Dynamical Systems*, dealing with the logistic family of real quadratic polynomials. The master student, Henrik Skjolding, made under supervision by Branner a careful numerical study of monic cubic polynomials, and afterwards Branner continued to iterate polynomials.

At the *Chaos* meeting, Branner asked Hubbard to tell her about the Hubbard trees. This became the starting point of a very fruitful collaboration. Hubbard convinced her easily that cubic polynomials were better treated when studied over the complex field. Moreover, if she wanted to shift to holomorphic dynamics there would be a great opportunity later that summer, when William Thurston would be lecturing on his groundbreaking topological characterization of rational functions at an NSF summer conference in Duluth.

Less than two months later Branner flew to the United States for the very first time and was introduced to an inspiring group of mathematicians working in dynamical systems, complex analysis, topology and differential geometry. After the conference, Branner continued with Hubbard to Cornell University. Already during the first week they became convinced that there was a wonderful structure in the cubic parameter space to be unfolded. Computer pictures generated by Homer Smith on the supercomputer at Cornell University supported this belief. As a result, Branner was invited as visiting professor to Cornell University, starting one year later.

Before then, in the summer of 1984, Douady, Hubbard and Branner spent some time at the Mittag-Leffler Institute in Sweden. Douady and Hubbard were most of the time working on the understanding of, and the filling in of details in, Thurston's proof

from the previous summer (where Douady had not been present), with Branner assigned ‘guinea pig’, taking notes while they were lecturing to each other, using the blackboard. They completed a joint preprint on Thurston’s theorem. Several years later the manuscript was published in Acta Mathematica. Thurstons theorem is a milestone in holomorphic dynamics. It asserts the equivalence of any un-obstructed post-critically finite branched self-cover of the sphere with hyperbolic orbifold to a unique rational map up to Möbius equivalence. In this theory the so-called Lattès maps play a special role (see also Milnor’s paper in this volume).

During the next one and a half year, while Branner was a visiting professor at Cornell University, Part I of [BH] on *The iteration of cubic polynomials* was finished. It describes the global topology of the parameter space $\mathbb{C}^2$ of monic, centered cubic polynomials of the form $P_{a,b}(z) = z^3 - a^2 z + b$. Several decompositions of the parameter space are considered. The first splitting is to separate the connectedness locus, where both critical points have bounded orbit and the Julia set therefore is connected, from the escape locus, where at least one critical point escapes to infinity. The second splitting is to foliate the escape locus into different hyper-surfaces, each one corresponding to a fixed maximal escape rate of the critical points. A particular way of constructing Teichmüller almost complex structures, invariant under $P_{a,b}$, were introduced as *wringing and stretching* the complex structure. This technique is now referred to as Branner-Hubbard motion (see also the three papers by Petersen & Tan, Tan and Douady in this volume). The wring and stretch operation is continuous on the cubic escape locus (it is not continuous on the entire cubic locus, see Tan in this volume). It follows that each

hyper-surface of fixed escape rate is homeomorphic to the three-dimensional sphere, and that the connectedness locus is cell-like: an infinite intersection of a nested sequence of closed topological three-dimensional balls. The third splitting is within each hyper-surface. Measuring the argument of the faster (first) escaping critical value (a choice of faster critical value if both escape at a common rate) one obtains a fibration over the circle with fibers a trefoil clover leaf. The final splitting in each clover leaf is governed by the behavior of the second critical point, which may escape or not. The structure of the set of parameters for which the second critical point does not escape can be described combinatorially. It includes infinitely many copies of the Mandelbrot set each with its own combinatorics. These stem from quadratic-like restrictions of (iterates of) cubic polynomials. In fact, the starting point of it all had been the understanding of this combinatorial structure.

In January 1986 Branner returned to Denmark. In Spring she learned from Douady that Yoccoz had observed that the combinatorial structure in a particular clover leaf over the zero-argument was similar to the combinatorial structure in the $\frac{1}{2}$-limb of the Mandelbrot set. How could this be justified? The comparison of quadratic polynomials with connected Julia set to cubic polynomials with disconnected Julia set was not obvious. However, moving along a stretching ray through such a cubic polynomial towards the connectedness locus the escape rate of the critical point decreases and in the limit the critical point is mapped onto the fixed point, which is the landing point of the zero-ray. In the cubic one-parameter family of polynomials with one critical point being pre-fixed, there is a limb corresponding to the fixed point being the landing point of the zero-ray. The correspondence between the post-critically finite quadratic and cubic polynomials can be understood in terms of Hubbard trees. It is much harder to establish a homeomorphism between the two relevant limbs.

Douady was at the time trying to embed the $\frac{1}{2}$-limb of the Mandelbrot set into the $\frac{1}{3}$-limb. He was motivated by the Corollary that would follow: The main vein in the $\frac{1}{3}$-limb is a topological arc, being the image of the line segment of the real axis included in the $\frac{1}{2}$-limb. It turned out that it was easier to work out the surgery technique that was needed to obtain the homeomorphisms between the relevant sets of quadratic and cubic polynomials than the one between the different sets of quadratic polynomials. Therefore, the surgery technique was developed in that order, and resulted in a joint paper [BD] *Surgery on Complex Polynomials* in Proceedings of Symposium on Dynamical Systems, Mexico 1986, with Theorem A relating the quadratic polynomials via a homeomorphism, and Theorem B relating the quadratic and cubic polynomials via a homeomorphism.

Part II of [BH] describing patterns and para-patterns was finished in the spring of 1988 during the special semester on dynamical systems at the Max-Planck-Institute für Mathematik in Bonn. The main inventions were on one hand the tableaus, combinatorial schemes, which catch enough information in the dynamical plane in order to estimate the modulus of annuli between consecutive critical levels of Green's function; and on the other hand the use of Grötzsch' inequality on moduli of annuli together with the result that for an open, bounded annulus of infinite modulus the bounded component of the complement is just one point. Hence, if for an infinite sequence of disjoint open annuli A_n, embedded in an open bounded annulus A, the infinite series of moduli of the A_n is divergent, then the annulus A has infinite modulus and surrounds exactly one point. This method of proving components to be point components has been called the divergence method. An infinite tree of patterns captures the structure in the dynamical planes. Out of these one builds para-patterns, which correspond to the structure in the clover leaves. The ends are either point components, as proved by the divergence method, or copies of the Mandelbrot set, corresponding to quadratic-like families.

The divergence method was later extended by Yoccoz to be applied to prove local connectivity of Julia sets of non-renormalizeable quadratic polynomials $z^2 + c$ with c in the Mandelbrot set and both fixed points repelling. He also proved local connectivity of the Mandelbrot set at the corresponding c-values. The complications in the quadratic setting is much more profound than in the cubic case, in particular the estimates in the parameter plane. Yoccoz called the division in the dynamical planes puzzles (instead of patterns) and the one in the parameter space para-puzzles (instead of para-patterns).

In the summer of 1993 Branner and one of the editors (PGH) organized a NATO advanced study institute in Hillerød, entitled *Real and Complex Dynamics*. For two weeks more than 100 participants, of which about two thirds were Ph.D.-students and Post. Docs., stayed together in ideal surroundings listening to lectures of 15 main speakers, combined with numerous talks by other participants and lots of informal discussions.

Much collaboration grew out of this summer school, and these young mathematicians are now leading the new developments. Branner began to work together with Nuria Fagella, further developing the surgery technique with the aim to prove certain symmetries in the Mandelbrot set, through comparisons with higher degree polynomials. Their first joint paper *Homeomorphisms between limbs of the Mandelbrot set* was finished in 1995 during the special semester on Conformal Dynamics at MSRI in Berkeley, arranged by Curt McMullen (see also the paper by Jung in this volume.)

10 years after the Hillerød meeting, in 2003, a large number of eminent researchers, colleagues young and old, gathered at the Søminestationen, a former Danish navy training site, now a peaceful conference centre picturesquely located by one of the many quiet fjords of the Danish coast. It was mid-Summer in Scandinavia. The symposium became a celebration of an exciting active area of mathematics, of warm and long lasting international friendships, and, not least, of the wonderful life and inspiring scholarship of Bodil Branner.

PGH & CLP

On Lattès Maps

John Milnor

Dedicated to Bodil Branner

Abstract. An exposition of the 1918 paper of Lattès, together with its historical antecedents, and its modern formulations and applications.

§1. The Lattès paper. In 1918, some months before his death of typhoid fever, Samuel Lattès published a brief paper describing an extremely interesting class of rational maps, which have since played a basic role as exceptional examples in the holomorphic dynamics literature. Similar examples had been described by Schröder almost fifty years earlier, but these seem to have been forgotten. (Compare §6.) In any case, Lattès provided a more general foundation for the study of these maps, and his name has become firmly attached to them.

His starting point was the "*Poincaré function*" $\theta : \mathbb{C} \to \widehat{\mathbb{C}}$ associated with a repelling fixed point $z_0 = f(z_0)$ of a rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. This can be described as the inverse of the Kœnigs linearization around z_0, extended to a globally defined meromorphic function.[1] Assuming for convenience that $z_0 \neq \infty$, it is characterized by the identity

$$f(\theta(t)) = \theta(\mu\, t)$$

for all complex numbers t, with $\theta(0) = z_0$, normalized by the condition that $\theta'(0) = 1$. Here $\mu = f'(z_0)$ is the *multiplier* at z_0, with $|\mu| > 1$. This Poincaré function can be computed explicitly by the formula

$$\theta(t) = \lim_{n\to\infty} f^{\circ n}(z_0 + t/\mu^n).$$

[1] Compare [La], [P], [K]. For general background material, see for example [M4] or [BM].

Its image $\theta(\mathbb{C}) \subset \widehat{\mathbb{C}}$ is equal to the Riemann sphere $\widehat{\mathbb{C}}$ with at most two points removed. In practice, we will always assume that f has degree at least two. The complement $\widehat{\mathbb{C}} \backslash \theta(\mathbb{C})$ is then precisely equal to the *exceptional set* $\mathcal{E}_f$, consisting of all points with finite grand orbit under f.

In general this Poincaré function θ has very complicated behavior. In particular, the Poincaré functions associated with different fixed points or periodic points are usually quite incompatible. However, Lattès pointed out that in special cases θ will be periodic or doubly periodic, and will give rise to a simultaneous linearization for all of the periodic points of f outside of the postcritical set. (For a more precise statement, see the proof of 3.9 below.)

We will expand on this idea in the following sections. Section 2 will introduce rational maps which are *finite quotients of affine maps*. (These are more commonly described in the literature as rational maps with *flat orbifold metric*—see §4.) They can be classified into *power maps*, *Chebyshev maps*, and *Lattès maps* according as the Julia set is a circle, a line or circle segment, or the entire Riemann sphere. These maps will be studied in Sections 3 through 5, concentrating on the Lattès case. Section 6 will describe the history of these ideas before Lattès; and §7 will describe some of the developments since his time. Finally, §8 will describe a number of concrete examples.

§2. Finite Quotients of Affine Maps. It will be convenient to make a very mild generalization of the Lattès construction, replacing the linear map $t \mapsto \mu\, t$ of his construction by an affine map $t \mapsto a\,t + b$. Let Λ be a discrete additive subgroup of the complex numbers $\mathbb{C}$. In the cases of interest, this subgroup will have rank either one or two, so that the quotient surface $\mathbb{C}/\Lambda$ is either a cylinder $\mathcal{C}$ or a torus $\mathcal{T}$.

Definition 2.1. A rational map f of degree two or more will be called a *finite quotient of an affine map* if there is a flat surface $\mathbb{C}/\Lambda$, an affine map $L(t) = a\,t + b$ from $\mathbb{C}/\Lambda$ to itself, and a finite-to-one holomorphic map $\Theta : \mathbb{C}/\Lambda \to \widehat{\mathbb{C}} \backslash \mathcal{E}_f$ which satisfies the semiconjugacy relation $f \circ \Theta = \Theta \circ L$. Thus the following diagram must commute:

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \stackrel{L}{\longrightarrow} & \mathbb{C}/\Lambda \\ \Theta\big\downarrow & & \Theta\big\downarrow \\ \widehat{\mathbb{C}} \backslash \mathcal{E}_f & \stackrel{f}{\longrightarrow} & \widehat{\mathbb{C}} \backslash \mathcal{E}_f . \end{array} \tag{1}$$

We can also write $f = \Theta \circ L \circ \Theta^{-1}$. It follows for example that any periodic orbit of L must map to a periodic orbit of f, and conversely that every periodic orbit of f outside of the exceptional set $\mathcal{E}_f$ is the image of a periodic orbit of L. (However, the periods are not necessarily the same.) Here the finite-to-one condition is essential. In fact it follows from Poincaré's construction that any rational map of degree at least two can be thought of as an infinite-to-one quotient of an affine map of $\mathbb{C}$.

These finite quotients of affine maps can be classified very roughly into three types, as follows. The set of postcritical points of f plays an important role in all cases. (Compare Lemma 3.4.)

Power maps. These are the simplest examples. By definition, a rational map will be called a *power map* if it is holomorphically conjugate to a map of the form

$$f_a(z) = z^a$$

where a is an integer. Note that f_a, restricted to the punctured plane $\mathbb{C}\backslash\{0\} = \widehat{\mathbb{C}}\backslash\mathcal{E}_{f_a}$, is conjugate to the linear map $t \mapsto at$ on the cylinder $\mathbb{C}/2\pi\mathbb{Z}$. In fact $f_a(e^{it}) = e^{i\,at}$, where the conjugacy $t \mapsto e^{it}$ maps $\mathbb{C}/2\pi\mathbb{Z}$ diffeomorphically onto $\mathbb{C}\backslash\{0\}$. The degree of f_a is equal to $|a|$, the Julia set $J(f_a)$ is equal to the unit circle, and the exceptional set $\mathcal{E}_{f_a} = \{0, \infty\}$ consists of the two critical points, which are also the two postcritical points.

Chebyshev maps. These are the next simplest examples. A rational map will be called a *Chebyshev map* if it is conjugate to $\pm\,\text{ч}_n(z)$ where ч_n is the degree n *Chebyshev polynomial*, defined by the equation[2]

$$\text{ч}_n(u + u^{-1}) = u^n + u^{-n}.$$

For example:

$$\text{ч}_2(z) = z^2 - 2, \qquad \text{ч}_3(z) = z^3 - 3z, \quad \text{ч}_4(z) = z^4 - 4z^2 + 2, \ldots .$$

We will see in §3.8 that power maps and Chebyshev maps are the only finite quotients of affine maps for which the lattice $\Lambda \subset \mathbb{C}$ has rank one.

If we set $u = e^{it}$, then the map $\Theta(t) = u + u^{-1} = 2\cos(t)$ is a proper map of degree two from the cylinder $\mathbb{C}/2\pi\mathbb{Z}$ to the plane $\mathbb{C}$, satisfying

$$\text{ч}_n(\Theta(t)) = \Theta(nt) \quad \text{or equivalently} \quad \text{ч}_n(2\cos t) = 2\cos(nt),$$

and also $-\text{ч}_n(\Theta(t)) = \Theta(nt + \pi)$. These identities show that both ч_n and $-\text{ч}_n$ are finite quotients of affine maps. The Julia set $J(\pm\,\text{ч}_n)$ is the closed interval $[-2, 2]$, and the exceptional set for $\pm\,\text{ч}_n$ is the singleton $\{\infty\}$. The postcritical set of $\pm\,\text{ч}_n$ consists of the three points $\{\pm 2, \infty\}$. In fact, if $2\cos(t)$ is a finite critical point of ч_n then by differentiating the equation $\text{ч}_n(2\cos t) = 2\cos(nt)$ we see that $\sin(nt) = 0$ and hence that $2\cos(nt) = \pm 2$.

Note: If n is even, the equation $-\text{ч}_n(z) = \text{ч}_n(kz)/k$ with $k = -1$ shows that $-\text{ч}_n$ is holomorphically conjugate to ч_n. However, for n odd the map $z \mapsto -\text{ч}_n(z)$ has a postcritical orbit $\{\pm 2\}$ of period two, and hence cannot be conjugate to $z \mapsto \text{ч}_n(z)$ which has only postcritical fixed points.

[2] The Russian letter ч is called "chi", pronounced as in "chicken".

Lattès maps. In the remaining case where the lattice $\Lambda \subset \mathbb{C}$ has rank two so that the quotient $\mathcal{T} = \mathbb{C}/\Lambda$ is a torus, the map $f = \Theta \circ L \circ \Theta^{-1}$ will be called a *Lattès map*. Here L is to be an affine self-map of the torus, and Θ is to be a holomorphic map from $\mathcal{T}$ to the Riemann sphere $\widehat{\mathbb{C}}$. These are the most interesting examples, and exhibit rather varied behavior. Thus we can distinguish between *flexible* Lattès maps which admit smooth deformations, and rigid Lattès maps which do not. (See 5.5 and 5.6, as well as §7 and 8.3.) Another important distinction is between the Lattès maps with three postcritical points, associated with triangle groups acting on the plane, and those with four postcritical points. (See §4.)

For any Lattès map f, since Θ is necessarily onto, there are no exceptional points. Furthermore, since periodic points of L are dense on the torus it follows that periodic points of f are dense on the Riemann sphere. Thus the Julia set $J(f)$ must be the entire sphere.

§3 Cyclic Group Actions on $\mathbb{C}/\Lambda$. The following result provides a more explicit description of all of the possible Lattès maps, as defined in §2.

Theorem 3.1. *A rational map is Lattès if and only if it is conformally conjugate to a map of the form $L/G_n : \mathcal{T}/G_n \to \mathcal{T}/G_n$ where:*

- *$\mathcal{T} \cong \mathbb{C}/\Lambda$ is a flat torus,*
- *G_n is the group of n-th roots of unity acting on $\mathcal{T}$ by rotation around a base point, with n equal to either* 2, 3, 4, *or* 6,
- *$\mathcal{T}/G_n$ is the quotient space provided with its natural structure as a smooth Riemann surface of genus zero* (*compare Remark* 3.6),
- *L is an affine map from $\mathcal{T}$ to itself which commutes with a generator of G_n, and*
- *L/G_n is the induced holomorphic map from the quotient surface to itself.*

Remark 3.2. The map $\mathcal{T} \to \mathcal{T}/G_n \cong \widehat{\mathbb{C}}$ can of course be described in terms of classical elliptic function theory. In the case $n = 2$ we can identify this map with the Weierstrass function $\wp : \mathbb{C}/\Lambda \to \widehat{\mathbb{C}}$ associated with the period lattice Λ. Here the lattice Λ or the torus $\mathcal{T}$ can be completely arbitrary, but in the cases $n \geq 3$ we will see that $\mathcal{T}$ is uniquely determined by n, up to conformal isomorphism. For $n = 3$ we can take the derivative $\wp'$ of the associated Weierstrass function as the semiconjugacy $\wp' : \mathcal{T} \to \widehat{\mathbb{C}}$, while for $n = 6$ we can use either $(\wp')^2$ or $\wp^3$ as the semiconjugacy. (For any lattice with G_3-symmetry, these two functions are related by the identity $(\wp')^2 = 4\wp^3 + \text{constant}$. The two alternate forms correspond to the fact that $\mathcal{T}/G_6$ can be identified either with $(\mathcal{T}/G_3)/G_2$ or with $(\mathcal{T}/G_2)/G_3$.) Finally, for $n = 4$ we can use the square $\wp^2$ of the associated Weierstrass function, corresponding to the factorization $\mathcal{T} \to \mathcal{T}/G_2 \to \mathcal{T}/G_4$.

Remark 3.3. This theorem is related to the definition in §2 as follows. Let us use the notation $\Theta^\star : \mathcal{T}^\star = \mathbb{C}/\Lambda^\star \to \widehat{\mathbb{C}} \backslash \mathcal{E}_f$ for the initial semiconjugacy of Definition 2.1, formula (1). The degree of this semiconjugacy $\Theta^\star$ can be arbitrarily large. However, the proof of 3.1 will show that $\Theta^\star$ can be factored in an essentially unique way as the composition of a covering map $\mathcal{T}^\star \to \mathcal{T}$ for some torus $\mathcal{T}$ and a projection map $\mathcal{T} \to \mathcal{T}/G_n \cong \widehat{\mathbb{C}}$ with n equal to 2, 3, 4, or 6.

The proof of 3.1 will be based on the following ideas. Let $\theta : \mathbb{C} \to \widehat{\mathbb{C}}$ be a doubly periodic meromorphic function, and let $\Lambda \subset \mathbb{C}$ be its lattice of periods so that $\lambda \in \Lambda$ if and only if $\theta(t+\lambda) = \theta(t)$ for all $t \in \mathbb{C}$. Then the canonical flat metric $|\mathbf{d}t|^2$ on $\mathbb{C}$ pushes forward to a corresponding flat metric on the torus $\mathcal{T} = \mathbb{C}/\Lambda$. If $\ell(t) = at + b$ is an affine map of $\mathbb{C}$ satisfying the identity $f \circ \theta = \theta \circ \ell$, then for $\lambda \in \Lambda$ and $t \in \mathbb{C}$ we have

$$\theta(at+b) = f(\theta(t)) = f(\theta(t+\lambda)) = \theta(a(t+\lambda)+b).$$

It follows that $a\Lambda \subset \Lambda$. Therefore the maps ℓ and θ on $\mathbb{C}$ induce corresponding maps L and Θ on $\mathcal{T}$, so that we have a commutative diagram of holomorphic maps

$$\begin{array}{ccc} \mathcal{T} & \stackrel{L}{\longrightarrow} & \mathcal{T} \\ \Theta\big\downarrow & & \Theta\big\downarrow \\ \widehat{\mathbb{C}} & \stackrel{f}{\longrightarrow} & \widehat{\mathbb{C}}. \end{array} \tag{2}$$

We will think of $\mathcal{T}$ as a branched covering of the Riemann sphere with projection map Θ. Since L carries a small region of area A to a region of area $|a|^2 A$, it follows that the map L has degree $|a|^2$. Using Diagram (2), we see that the degree d_f of the map f must also be equal to $|a|^2$. We will always assume that $d_f \geq 2$.

One easily derived property is the following. (For a more precise statement, see 4.5.) Let C_f be the set of critical points of f and let $V_f = f(C_f)$ be the set of critical values. Similarly, let $V_\Theta = \Theta(C_\Theta)$ be the set of critical values for the projection map Θ.

Lemma 3.4. *Every Lattès map f is postcritically finite. In fact the postcritical set*

$$P_f = V_f \cup f(V_f) \cup f^{\circ 2}(V_f) \cup \cdots$$

for f is precisely equal to the finite set V_Θ consisting of all critical values for the projection $\Theta : \mathcal{T} \to \widehat{\mathbb{C}}$.

Proof. Let $d_f(z)$ be the local degree of the map f at a point z. Thus

$$1 \leq d_f(z) \leq d_f,$$

where $d_f(z) > 1$ if and only if z is a critical point of f. Given points $\tau_j \in \mathcal{T}$ and $z_j \in \widehat{\mathbb{C}}$ with

$$\begin{array}{ccc} \tau_1 & \stackrel{L}{\mapsto} & \tau_0 \\ \Theta\downarrow & & \Theta\downarrow \\ z_1 & \stackrel{f}{\mapsto} & z_0, \end{array}$$

since L has local degree $d_L(\tau) = 1$ everywhere, it follows that

$$d_\Theta(\tau_0) = d_\Theta(\tau_1) \cdot d_f(z_1). \tag{3}$$

Since the maps L and Θ are surjective, it follows that z_0 is a critical value of Θ if and and if it is either a critical value of f or has a preimage $z_1 \in f^{-1}(z_0)$ which is a critical value of Θ or both. Thus $V_\Theta = V_f \cup f(V_\Theta)$, which implies inductively that

$$f^{\circ n}(V_f) \subset V_\Theta, \qquad \text{hence} \qquad P_f \subset V_\Theta.$$

On the other hand, if some critical point τ_0 of Θ had image $\Theta(\tau_0)$ outside of the post-critical set P_f, then all of the infinitely many iterated preimages $\cdots \mapsto \tau_2 \mapsto \tau_1 \mapsto \tau_0$ would have the same property. This is impossible, since Θ can have only finitely many critical points. □

We will prove the following preliminary version of 3.1, with notations as in Diagram (2).

Lemma 3.5. *If f is a Lattès map, then there is a finite cyclic group G of rigid rotations of the torus $\mathcal{T}$ about some base point, so that $\Theta(\tau') = \Theta(\tau)$ if and only if $\tau' = g\,\tau$ for some $g \in G$. Thus Θ induces a canonical homeomorphism from the quotient space $\mathcal{T}/G$ onto the Riemann sphere.*

Remark 3.6. Such a quotient $\mathcal{T}/G$ can be given two different structures which are distinct, but closely related. Suppose that a point $\tau_0 \in \mathcal{T}$ is mapped to itself by a non-trivial subgroup of G, necessarily cyclic of order $r > 1$. Any τ close to τ_0 can be written as the sum of τ_0 with a small complex number $\tau - \tau_0$. The power $(\tau - \tau_0)^r$ then serves as a local uniformizing parameter for $\mathcal{T}/G$ near τ_0. In this way, the quotient becomes a smooth Riemann surface. On the other hand, if we want to carry the flat Euclidean structure of $\mathcal{T}$ over to $\mathcal{T}/G$, then the image of τ_0 must be considered as a singular "cone point", as described in the next section. The integer r, equal to the local degree $d_\Theta(\tau_0)$, depends only on $\Theta(\tau_0)$, and is called the *ramification index* of $\Theta(\tau_0)$. (See 4.1.)

Proof of 3.5. Let U be any simply connected open subset of $\widehat{\mathbb{C}} \setminus P_f = \widehat{\mathbb{C}} \setminus V_\Theta$. Then the preimage $\Theta^{-1}(U)$ is the union $U_1 \cup \cdots \cup U_n$ of n disjoint open sets, each of which projects diffeomorphically onto U, where $n = d_\Theta$ is equal to the degree of Θ. Let $\Theta_j : U_j \xrightarrow{\cong} U$ be the restriction of Θ to U_j. We will first prove that each composition

$$\Theta_k^{-1} \circ \Theta_j : U_j \to U_k, \tag{4}$$

is an isometry from U_j onto U_k, using the standard flat metric on the torus.

Since periodic points of f are everywhere dense, we can choose a periodic point $z_0 \in U$. Now replacing f by some carefully chosen iterate $\widehat{f}$, and replacing L by the corresponding iterate $\widehat{L}$, we may assume without changing Θ that:

- z_0 is actually a fixed point of $\widehat{f}$, and that
- every point in the finite $\widehat{L}$-invariant set $\Theta^{-1}(z_0)$ is either fixed by $\widehat{L}$, or is mapped to a fixed point by $\widehat{L}$.

In other words, each point $\tau_j = \Theta_j^{-1}(z_0)$ is either a fixed point of $\widehat{L}$ or maps directly to a fixed point. For τ close to τ_j, evidently the difference $\tau - \tau_j$ can be identified with a unique complex number close to zero. Setting $\widehat{L}(\tau_j) = \tau_{j'}$, note that the affine map

$$\tau - \tau_j \mapsto \widehat{L}(\tau) - \tau_{j'} \in \mathbb{C}$$

is actually linear, so that $\widehat{L}(\tau) - \tau_{j'} = \widehat{a}\,(\tau - \tau_j)$ where $\widehat{a} = \widehat{L}'$ is constant. Similarly, for z close to z_0, the difference

$$K_j(z) = \Theta_j^{-1}(z) - \tau_j$$

is well defined as a complex number. For each index j we will show that the map $z \mapsto K_j(z) \in \mathbb{C}$ is a Kœnigs linearizing map for $\widehat{f}$ in a neighborhood of z_0. That is,

$$K_j(\widehat{f}(z)) = \widehat{a}\,K_j(z), \qquad \text{with} \qquad K_j(z_0) = 0, \tag{5}$$

where the constant $\widehat{a} = \widehat{L}'$ is necessarily equal to the multiplier of $\widehat{f}$ at z_0. In fact the identity $\Theta_{j'}^{-1}(\widehat{f}(z)) = \widehat{L}(\Theta_j^{-1}(z))$ holds for all z close to z_0. Subtracting $\tau_{j'}$ we see that

$$K_{j'}(\widehat{f}(z)) = \widehat{a}\,K_j(z). \tag{6}$$

If τ_j is a fixed point so that $j' = j$, then this is the required assertion (5). But $\tau_{j'}$ is always a fixed point, so this proves that $K_{j'}(\widehat{f}(z)) = \widehat{a}\,K_{j'}(z)$. Combining this equation with (6), we see that $K_j(z) = K_{j'}(z)$, and it follows that equation (5) holds in all cases.

Since such a Kœnigs linearizing map is unique up to multiplication by a constant, it follows that every $K_i(z)$ must be equal to the product $c_{ij}\,K_j(z)$ for some constant $c_{ij} \neq 0$ and for all z close to z_0. Therefore $\Theta_i^{-1}(z)$ must be equal to $c_{ij}\,\Theta_j^{-1}(z)$ plus a constant for all z close to z_0. Choosing a local lifting of $\Theta_i^{-1} \circ \Theta_j$ to the universal covering space $\widetilde{\mathcal{T}} \cong \mathbb{C}$ and continuing analytically, we obtain an affine map A_{ij} from $\mathbb{C}$ to itself with derivative $A'_{ij} = c_{ij}$, satisfying the identity $\theta = \theta \circ A_{ij}$, where θ is the composition $\widetilde{\mathcal{T}} \to \mathcal{T} \xrightarrow{\Theta} \widehat{\mathbb{C}}$.

We must prove that $|c_{jk}| = 1$, so that this affine transformation is an isometry. Let $\widetilde{G}$ be the group[3] consisting of all affine transformations $\widetilde{g}$ of $\mathbb{C}$ which satisfy the identity $\theta = \theta \circ \widetilde{g}$. The translations $t \mapsto t + \lambda$ with $\lambda \in \Lambda$ constitute a normal subgroup, and the quotient $G = \widetilde{G}/\Lambda$ acts as a finite group of complex affine automorphisms of the torus $\mathcal{T} = \mathbb{C}/\Lambda$. In fact G has exactly n elements, since it contains exactly one transformation g carrying U_1 to any specified U_j. The derivative map $g \mapsto g'$ is an injective homomorphism from G to the multiplicative group $\mathbb{C} \setminus \{0\}$. Hence it must carry G isomorphically onto the unique subgroup of $\mathbb{C} \setminus \{0\}$ of order n, namely the group G_n of n-th roots of unity. Furthermore, a generator of G must have a fixed point in the torus, so G can be considered as a group of rotations about this fixed point. This completes the proof of 3.5. □

In fact, if we translate coordinates so that some specified fixed point of the G-action is the origin of the torus $\mathcal{T} = \mathbb{C}/\Lambda$, then clearly we can identify G with the group G_n of n-th roots of unity, acting by multiplication on $\mathcal{T}$.

[3] This $\widetilde{G}$ is often described as a ***crystallographic group*** acting on $\mathbb{C}$; that is, it is a discrete group of rigid Euclidean motions of $\mathbb{C}$, with compact quotient $\mathbb{C}/\widetilde{G} \cong \mathcal{T}/G$. (Compare 4.9 and 7.3.)

Lemma 3.7. *The order n of such a cyclic group of rotations of the torus with quotient $\mathcal{T}/G_n \cong \widehat{\mathbb{C}}$ is necessarily either* 2, 3, 4, *or* 6.

Proof. Thinking of a rotation through angle α as a real linear map, it has eigenvalues $e^{\pm i\alpha}$ and trace $e^{i\alpha} + e^{-i\alpha} = 2\cos(\alpha)$. On the other hand, if such a rotation carries the lattice Λ into itself, then its trace must be an integer. The function $\alpha \mapsto 2\cos(\alpha)$ is monotone decreasing for $0 < \alpha \le \pi$ and takes only the following integer values:

$$\begin{array}{rcrrrr} r & = & 6 & 4 & 3 & 2 \\ 2\cos(2\pi/r) & = & 1 & 0 & -1 & -2. \end{array}$$

This proves 3.7. □

Now to complete the proof of Theorem 3.1, we must find which affine maps $L(\tau) = a\,\tau + b$ give rise to well defined maps of the quotient surface $\mathcal{T}/G_n$. Let ω be a primitive n-th root of unity, so that the rotation $g(t) = \omega t$ generates G_n. Then evidently the points $L(t) = at + b$ and $L(g(t)) = a\,\omega\,t + b$ represent the same element of $\mathcal{T}/G_n$ if and only if

$$a\,\omega\,t + b \equiv \omega^k(at + b) \bmod \Lambda \quad \text{for some power} \quad \omega^k.$$

If this equation is true for some generic choice of t, then it will be true identically for all t. Now differentiating with respect to t we see that $\omega^k = \omega$, and substituting $t = 0$ we see that $b \equiv \omega\, b \bmod \Lambda$. It follows easily that $g \circ L = L \circ g$. Conversely, whenever g and L commute it follows immediately that L/G_n is well defined. This completes the proof of 3.1.

The analogous statement for Chebyshev maps and power maps is the following.

Lemma 3.8. *If $f = \Theta \circ L \circ \Theta^{-1}$ is a finite quotient of an affine map on a cylinder $\mathcal{C}$, then f is holomorphically conjugate either to a power map $z \mapsto z^a$ or to a Chebyshev map $\pm\,\text{ų}_d$.*

The proof is completely analogous to the proof of 3.1. In fact any such f is conjugate to a map of the form $L/G_n : \mathcal{C}/G_n \to \mathcal{C}/G_n$, where L is an affine map of the cylinder $\mathcal{C}$ and where n is either one (for the power map case) or two (for the Chebyshev case). Details will be left to the reader. □

The following helps to demonstrate the extremely restricted dynamics associated with finite quotients of affine maps. Presumably nothing like it is true for any other rational map.

Corollary 3.9. Let $f = \Theta \circ L \circ \Theta^{-1}$ *be a finite quotient of an affine* map L *which has derivative $L' = a$. If $z \in \widehat{\mathbb{C}} \setminus \mathcal{E}_f$ is a periodic point with period $p \ge 1$ and ramification index $r \ge 1$, then the multiplier of $f^{\circ p}$ at z is a number of the form $\mu = (\omega a^p)^r$ where $\omega^n = 1$.*

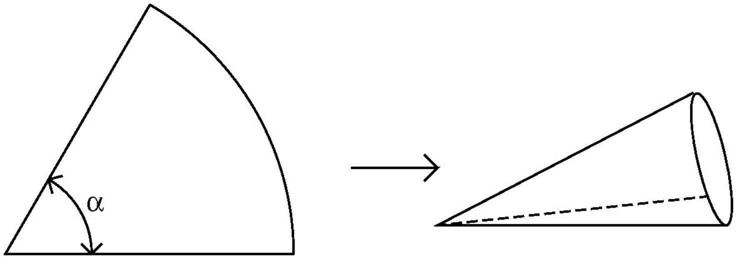


Figure 1: Model for a cone point with cone angle α.

(The ramification index is described in 3.6 and also in §4.) For example for a periodic orbit of maximal ramification index $r = n$ the multiplier is simply a^{pn}. In the case of a generic periodic orbit with $r = 1$, the multiplier has the form ωa^p. In all cases, the absolute value $|\mu|$ is equal to $|a|^{pr}$.

Proof of 3.9. First consider a fixed point $z_0 = f(z_0)$ and let $\Theta(\tau_0) = z_0$. As in 3.6, we can take $\zeta = (\tau - \tau_0)^r$ as local uniformizing parameter near z_0. On the other hand, since $z_0 = f(z_0)$ we have $\tau_0 \sim L(\tau_0)$, or in other words $\tau_0 = \omega\, L(\tau_0)$ for some $\omega \in G_n$. Thus f lifts to the linear map

$$\tau - \tau_0 \mapsto \omega L(\tau) - \omega L(\tau_0) = \omega\, a\, (\tau - \tau_0).$$

Therefore, in terms of the local coordinate ζ near z_0, we have the linear map $\zeta \mapsto (\omega\, a)^r \zeta$, with derivative $\mu = (\omega\, a)^r$. Applying the same argument to the p-th iterates of f and L, we get a corresponding identity for a period p orbit. □

§4. Flat Orbifold Metrics. We can give another characterization of finite quotients of affine maps as follows. Recall that $\mathcal{E}_f$ is the exceptional set for f, with at most two points.

Definition. A *flat orbifold metric* on $\widehat{\mathbb{C}} \setminus \mathcal{E}_f$ is a complete metric which is smooth, conformal, and locally isometric to the standard flat metric on $\mathbb{C}$, except at finitely many "cone points", where it has cone angle of the form $2\pi/r$. Here a *cone point* with *cone angle* $0 < \alpha < 2\pi$, is an isolated singular point of the metric which can be visualized by cutting an angle of α out of a sheet of paper and then gluing the two edges together. (Compare Figure 1. A more formal definition will be left to the reader.) In the special case where α is an angle of the form $2\pi/r$, we can identify such a cone with the quotient space $\mathbb{C}/G_r$ where G_r is the group of r-th roots of unity acting by multiplication on the complex numbers, and where the flat metric on $\mathbb{C}$ corresponds to a flat metric on the quotient, except at the cone point.

Evidently the canonical flat metric on a torus $\mathcal{T}$ or cylinder $\mathcal{C}$ induces a corresponding flat orbifold metric on the quotient $\mathcal{T}/G_n$ of Theorem 3.1 or the quotient $\mathcal{C}/G_n$ of

Lemma 3.8. Thus near any non-cone point we can choose a local coordinate t so that the metric takes the form $|\mathbf{d}t|^2$. I will say that such a metric *linearizes* the map f since, in terms of such preferred local coordinates, f is an affine map with constant derivative.[4] (An equivalent property is that f maps any curve of length δ to a curve of length $k\,\delta$ where $k = |a| > 1$ is constant.) A converse statement is also true:

Theorem 4.1. *A rational* map f *is a finite quotient of an affine map if and only if it is linearized by some flat orbifold metric, or if and only if there exists an integer valued "ramification index" function* $r(z)$ *on* $\widehat{\mathbb{C}}\backslash\mathcal{E}_f$ *satisfying the identity*

$$r(f(z)) = d_f(z)\; r(z) \quad \textit{for all } z, \tag{7}$$

with $r(z) = 1$ *outside of the postcritical set of* f.

Proof in the Lattès case. First suppose that f is a finite quotient of an affine map on a torus, conformally conjugate to the quotient map $L/G_n : \mathcal{T}/G_n \to \mathcal{T}/G_n$. If τ_0 is a critical point of the projection $\Theta : \mathcal{T} \to \mathcal{T}/G_n \cong \widehat{\mathbb{C}}$ with local degree $d_\Theta(\tau_0) = r$, then the subgroup consisting of elements of G_n which fix τ_0 must be generated by a rotation through $2\pi/r$ about τ_0. Hence the flat metric on $\mathcal{T}$ pushes forward to a flat metric on $\mathcal{T}/G_n$ with τ_0 corresponding to a cone point $z_0 = \Theta(\tau_0)$ of angle $2\pi/r$. This integer $r = r(z_0) > 1$ is called the ramification index of the cone point. Setting $r(z) = 1$ if z is not a cone point, we see that $r(\Theta(\tau))$ can be identified with the local degree $d_\Theta(\tau)$ in all cases. There may be several different points in $\Theta^{-1}(z)$, but Θ must have the same local degree at all of these points, since the angle at a cone point is uniquely defined, or by 3.9. With these notations, the required equation (7) is just a restatement of equation (3) of §3.

Conversely, suppose that (7) is satisfied. *It follows from this equation that* f *is postcritically finite.* In fact we can express the postcritical set P_f as a union $P_1 \cup P_2 \cup \cdots$ of disjoint finite subsets, where

$$P_1 = f(C_f) \quad \text{and} \quad P_{m+1} = f(P_m)\backslash(P_1 \cup \cdots \cup P_m).$$

Let $|P_m|$ be the number of elements in P_m. Since $f(P_m) \supset P_{m+1}$, the sequence $|C_f| \geq |P_1| \geq |P_2| \geq |P_3| \geq \cdots$ must eventually stabilize. Therefore we can choose an integer m so that P_k maps bijectively onto P_{k+1} for $k \geq m$, and must prove that the number of elements $|P_m| = |P_{m+1}| = \cdots$ is zero. Note that each point of P_{m+1} has d_f distinct preimages, where $d_f \geq 2$ by our standing hypothesis. Thus if $|P_{m+1}| \neq 0$ there would exist some point $z \notin P_m$ with $f(z) \in P_{m+1}$. In fact it would follow that $z \notin C_f \cup P_f$. For if z were in $C_f \cup P_1 \cup \cdots \cup P_{m-1}$ then $f(z)$ would be in some P_k with $k < m+1$, while if z were in P_k with $k > m$ then $f(z)$ would be in P_{k+1}

[4] The more usual terminology for a map f which is linearized by a flat orbifold metric would be that f has "*parabolic*" or "*Euclidean*" *orbifold.* Following Thurston, for any postcritically finite f there is a smallest function $r(z) \geq 1$ on $\widehat{\mathbb{C}}\backslash\mathcal{E}_f$ such that $r(f(z))$ is a multiple of $d_f(z)\,r(z)$ for every z. Furthermore there is an essentially unique complete orbifold metric of constant curvature ≤ 0 on $\widehat{\mathbb{C}}\backslash\mathcal{E}_f$ with $r(z)$ as ramification index function. The curvature is zero if and only if equation (7) of 4.1 is satisfied. (See [DH] or [M4].)

with $k+1 > m+1$, contradicting the hypothesis that $f(z) \in P_{m+1}$ in either case. The existence of a point $z \notin C_f \cup P_f$ with $f(z) \in P_f$ clearly contradicts equation (7).

Let me use the notation M for the Riemann sphere $\widehat{\mathbb{C}}$ together with the "orbifold structure" determined by the function $r : \widehat{\mathbb{C}} \to \{1, 2, 3, \ldots\}$. The *universal covering orbifold* $\widetilde{M}$ can be characterized as a simply connected Riemann surface together with a holomorphic branched covering map $\theta : \widetilde{M} \to M = \widehat{\mathbb{C}}$ such that, for each $\widetilde{z} \in \widetilde{M}$, the local degree $d_\theta(\widetilde{z})$ is equal to the prescribed ramification index $r(\theta(\widetilde{z}))$. Such a universal covering associated with a function $r : M \to \{1, 2, 3, \ldots\}$ exists whenever the number of z with $r(z) > 1$ is finite with at least three elements. (See for example [M4, Lemma E.1].)

For this proof only, it will be convenient to choose some fixed point of f as base point $z_0 \in M$. Using equation (7), there is no obstruction to lifting f to a holomorphic map $\widetilde{f}$ which maps the Riemann surface $\widetilde{M}$ diffeomorphically into itself, with no critical points. Furthermore, we can choose $\widetilde{f}$ to fix some base point $\widetilde{z}_0$ lying over z_0. The covering manifold $\widetilde{M}$ cannot be a compact surface, necessarily of genus zero, since then $\widetilde{f}$ and hence f would have degree one, contrary to the standing hypothesis that $d_f \geq 2$. Furthermore, since the base point in $\widetilde{M}$ is strictly repelling under $\widetilde{f}$, it follows that $\widetilde{M}$ cannot be a hyperbolic surface. Therefore $\widetilde{M}$ must be conformally isomorphic to the complex numbers $\mathbb{C}$, and $\widetilde{f}$ must correspond to a linear map L from $\mathbb{C}$ to itself. Evidently the standard flat metric on $\mathbb{C} \cong \widetilde{M}$ now gives rise to a flat orbifold metric on $M = \widehat{\mathbb{C}}$ which linearizes the map f.

Finally, suppose that we start with a flat orbifold metric on $\widehat{\mathbb{C}}$ which linearizes the rational map f. The preceding discussion shows that f lifts to a linear map $\widetilde{f}$ on the universal covering orbifold $\widetilde{M}$. Let $\widetilde{G}$ be the group of *deck transformations* of $\widetilde{M}$, that is homeomorphisms $\widetilde{g}$ from $\widetilde{M}$ to itself which cover the identity map of M, so that $\theta \circ \widetilde{g} = \theta$. Then the quotient surface $\widetilde{M}/\widetilde{G}$ can be identified with $M = \widehat{\mathbb{C}}$. If $\Lambda \subset \widetilde{G}$ is the normal subgroup consisting of those deck transformations which are translations of $\widetilde{M} \cong \mathbb{C}$, then the quotient group $G = \widetilde{G}/\Lambda$ is a finite group of rotations with order equal to the least common multiple of the ramification indices. It follows that the quotient $\mathcal{T} = \widetilde{M}/\Lambda$ is a torus, and hence that f is a finite quotient of an affine map of this torus.

The proof of 4.1 in the Chebyshev and power map cases is similar and will be omitted. □

Remark 4.2. Note that the construction of the torus $\mathcal{T}$, the group G_n and the affine map L from the rational map f satisfying (7) is completely canonical, except for the choice of lifting for f. For example, when there are four postcritical points, the conformal conjugacy class of the torus $\mathcal{T}$ is completely determined by the set of postcritical points, and in fact by the cross-ratio of these four points.

However, to make an explicit classification we must note the following.

• We want to identify the torus $\mathcal{T}$ with some quotient $\mathbb{C}/\Lambda$. Here, Λ is unique only up to multiplication by a non-zero constant; but we can make an explicit and unique choice by taking Λ to be the lattice $\mathbb{Z} \oplus \gamma\mathbb{Z}$ spanned by 1 and γ, where γ belongs to the Siegel region

$$|\gamma| \geq 1, \quad |\Re(\gamma)| \leq 1/2, \quad \Im(\gamma) > 0, \\ \text{with} \quad \Re(\gamma) \geq 0 \text{ whenever } |\gamma| = 1 \text{ or } |\Re(\gamma)| = 1/2. \tag{8}$$

With these conditions, γ is uniquely determined by the conformal isomorphism class of $\mathcal{T}$. We will describe the corresponding $\Lambda = \mathbb{Z} \oplus \gamma\mathbb{Z}$ as a *normalized lattice*.

• For specified Λ, we still need to make some choice of conformal isomorphism $\upsilon : \mathbb{C}/\Lambda \to \mathcal{T}$. In most cases, υ depends only on a choice of base point $\upsilon(0) \in \mathcal{T}$, up to sign. However, in the special case where $\mathcal{T}$ admits a G_3 (or G_4) action, we can also multiply υ by a cube (or fourth) root of unity. As in §3, it will be convenient to choose one of the fixed points of the G_n action as a base point in $\mathcal{T}$.

• The lifting $L(t) = at + b$ of the map f to the torus is well defined only up to the action of G_n. In particular, we are always free to multiply the coefficients a by an n-th root of unity.

We will deal with all of these ambiguities in §5.

Here is an interesting consequence of 4.1. Let f and g be rational maps.

Corollary 4.3. *Suppose that there is a holomorphic semiconjugacy from f to g, that is, a non-constant rational map h with $h \circ f = g \circ h$. Then f is a finite quotient of an affine map if and only if g is a finite quotient of an affine map.*

Proof. It is not hard to see that $h^{-1}(\mathcal{E}_g) = \mathcal{E}_f$, so that h induces a proper map from $\widehat{\mathbb{C}} \backslash \mathcal{E}_f$ to $\widehat{\mathbb{C}} \backslash \mathcal{E}_g$. Now if f is a finite quotient of an affine map L, say $f = \Theta \circ L \circ \Theta^{-1}$, then it follows immediately that $g = (h \circ \Theta) \circ L \circ (h \circ \Theta)^{-1}$. Conversely, if g is such a finite quotient, then there is a flat orbifold structure on $\widehat{\mathbb{C}} \backslash \mathcal{E}_g$ which linearizes g, and we can lift easily to a flat orbifold structure on $\widehat{\mathbb{C}} \backslash \mathcal{E}_f$ which linearizes f. □

In order to classify all possible flat orbifold structures on the Riemann sphere, we can use a piecewise linear form of the Gauss-Bonnet Theorem. For this lemma only, we allow cone angles which are greater than 2π.

Lemma 4.4. *If a flat metric with finitely many cone points on a compact Riemann surface S has cone angles $\alpha_1, \ldots, \alpha_k$, then*

$$(2\pi - \alpha_1) + \cdots + (2\pi - \alpha_k) = 2\pi\, \chi(S), \tag{9}$$

where $\chi(S)$ is the Euler characteristic. In particular, if $\alpha_j = 2\pi/r_j$ and if S is the Riemann sphere with $\chi(S) = 2$, then it follows that $\sum (1 - 1/r_j) = 2$.

Proof. Choose a rectilinear triangulation, where the cone points will necessarily be among the vertices. Let V be the number of vertices, E the number of edges, and F the number of faces (i.e., triangles). Then $2E = 3F$ since each edge bounds two triangles and each triangle has three edges. Thus

$$\chi(S) = V - E + F = V - F/2. \tag{10}$$

The sum of the internal angles of all of the triangles is clearly equal to πF. On the other hand, the j-th cone point contributes α_j to the total, while each non-cone vertex contributes 2π. Thus

$$\pi F = \alpha_1 + \cdots + \alpha_k + 2\pi(V - k). \tag{11}$$

Multiplying equation (10) by 2π and using (11), we obtain the required equation (9). □

Corollary 4.5. *The collection of ramification indices for a flat orbifold metric on the Riemann sphere must be either* $\{2, 2, 2, 2\}$ *or* $\{3, 3, 3\}$ *or* $\{2, 4, 4\}$ *or* $\{2, 3, 6\}$. *In particular, the number of cone points must be either four or three.*

Proof. Using the inequality $1/2 \le (1 - 1/r_j) < 1$, it is easy to check that the required equation

$$\sum_j (1 - 1/r_j) = \chi(\widehat{\mathbb{C}}) = 2,$$

has only these solutions in integers $r_j > 1$. □

Remark 4.6. If $z \in \widehat{\mathbb{C}}$ corresponds to a fixed point for the action of the group G_n on the torus, then the ramification index $r(z)$ is evidently equal to n. For any other point, it is some divisor of n. Thus the order n of the rotation group G_n can be identified with the least common multiple (or the maximum) of the various ramification indices as listed in 4.5.

Remark 4.7. To deal with the case of a map f which has exceptional points, we can assign the ramification index $r(z) = \infty$ to any exceptional point $z \in \mathcal{E}_f$. If we allow such points, then the equation $\sum (1 - 1/r_j) = 2$ has two further solutions, namely: $\{\infty,\ \infty\}$ corresponding to the power map case, and $\{2,\ 2,\ \infty\}$ corresponding to the Chebyshev case.

Combining Corollary 4.5 with equation (7), we get an easy characterization of Lattès maps in two of the four cases.

Corollary 4.8. *A rational map with four postcritical points is Lattès if and only if every critical point is simple* (*with local degree two*) *and no critical point is postcritical. Similarly, a rational map with three postcritical points is Lattès of type* $\{3, 3, 3\}$ *if and only if every critical point has local degree three and none is postcritical.*

The proof is easily supplied. □

Remark 4.9. We conclude this section with a more precise description of the possible crystallographic groups $\widetilde{G}$ acting on $\mathbb{C}$, and of the corresponding orbifold geometries on $\mathbb{C}/\widetilde{G} \cong \mathcal{T}/G_n$. We first look at the cases $n \geq 3$ where there are exactly three cone points in $\mathcal{T}/G_n$ or equivalently three postcritical points for any associated Lattès map. Thus the collection of ramification indices must be either $\{2, 3, 6\}$ or $\{2, 4, 4\}$ or $\{3, 3, 3\}$. Each of these three possibilities is associated with a rigidly defined flat orbifold geometry which can be described as follows. Join each pair of cone points by a minimal geodesic. It is not hard to check that these three geodesics cannot cross each other; and no geodesic can pass through a cone point since our cone angles are strictly less than 2π. In this way, we obtain three edges which cut our locally flat manifold into two Euclidean triangles. Since these two triangles have the same edges, they must be precise mirror images of each other. In particular, the two edges which meet at a cone point of angle $2\pi/r_j$ must cut it into two Euclidean angles of π/r_j. Passing to the branched covering space $\mathcal{T}$ or its universal covering $\widetilde{\mathcal{T}}$, we obtain a tiling of the torus or the Euclidean plane[5] by triangles with angles π/r_1, π/r_2 and π/r_3. These tilings are illustrated in Figures 2, 3, 4.

In each case, each pair of adjacent triangles are mirror images of each other, and together form a fundamental domain for the action of the group of Euclidean motions $\widetilde{G}_n$ on the plane, or for the action of G_n on the torus. For each vertex of this diagram, corresponding to a cone point of angle $2\pi/r_j$, there are r_j lines through the vertex, and hence $2r_j$ triangles which meet at the vertex. The subgroup of $\widetilde{G}_n$ (or G_n) which fixes such a point has order r_j and is generated by a rotation through the angle $\alpha_i = 2\pi/r_j$.

The subgroup $\Lambda \subset \widetilde{G}_n$ consists of all translations of the plane which belong to $\widetilde{G}_n$. Recall from 4.4 that the integer n can be described as the maximum of the r_j. The $2n$ triangles which meet at any maximally ramified vertex form a fundamental domain for the action of this subgroup Λ. In the $\{2, 3, 6\}$ and $\{3, 3, 3\}$ cases, this fundamental domain is a regular hexagon, while in the $\{2, 4, 4\}$ case it is a square. In all three cases, the torus $\mathcal{T}$ can be obtained by identifying opposite faces of this fundamental domain under the appropriate translations. Thus when $n \geq 3$ the torus $\mathcal{T}$ is uniquely determined by n, up to conformal diffeomorphism.

In the $\{2, 3, 6\}$ case, the integers r_j are all distinct, so it is easy to distinguish the three kinds of vertices. However, in the $\{2, 4, 4\}$ case there are two different kinds of vertices of index 4. In order to distinguish them, one kind has been marked with dots and the other with circles. Similarly in the $\{3, 3, 3\}$ case, the three kinds of vertices have been marked in three different ways. In this last case, half of the triangles have also been labeled. In all three cases, the points of the lattice Λ, corresponding to the base point in $\mathcal{T}$, have been circled. For all three diagrams, the group $\widetilde{G}_n$ can be described as the group of all rigid Euclidean motions which carry the marked diagram

[5] More generally, for any triple of integers $r_j \geq 2$ there is an associated tiling, either of the Euclidean or hyperbolic plane or of the 2-sphere depending on the sign of $1/r_1 + 1/r_2 + 1/r_3 - 1$. See for example [M1].

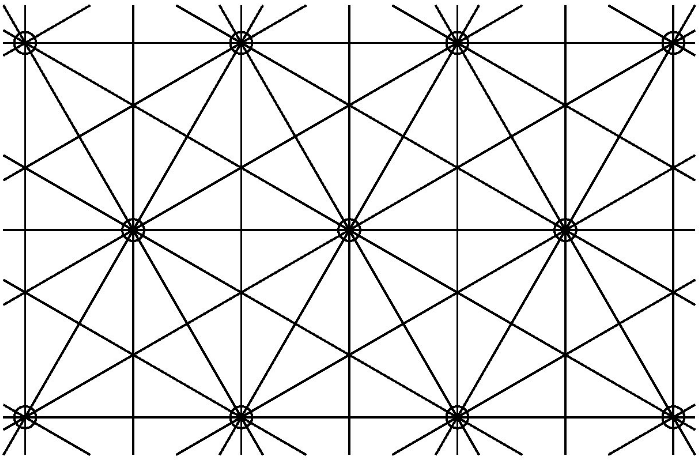


Figure 2: The $\{2,3,6\}$-tiling of the plane. In each of these diagrams, all of the points of maximal ramification n have been marked.

to itself, and the lattice Λ can be identified with the subgroup consisting of translations which carry this marked diagram to itself.

The analogue of Figures 2, 3, 4 for a typical orbifold of type $\{2, 2, 2, 2\}$ is a tiling of the plane by parallelograms associated with a typical lattice $\Lambda = \mathbb{Z} \oplus \gamma\mathbb{Z}$, as

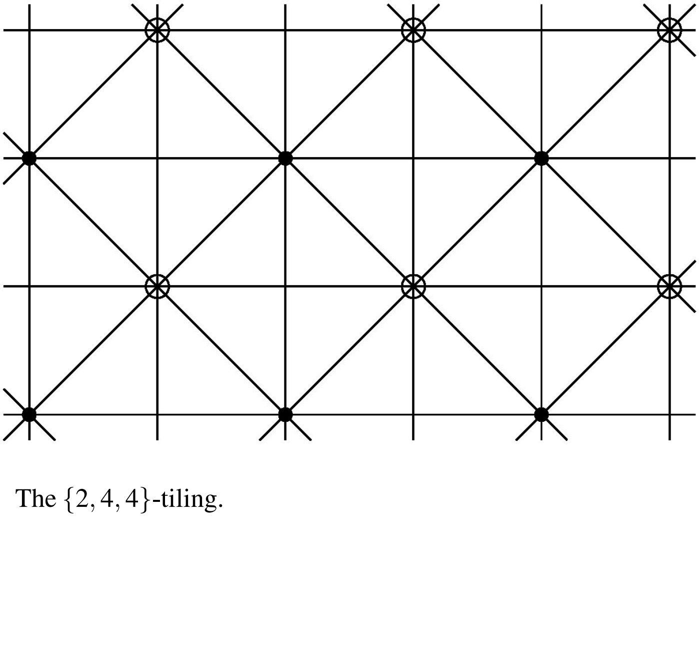


Figure 3: The $\{2,4,4\}$-tiling.

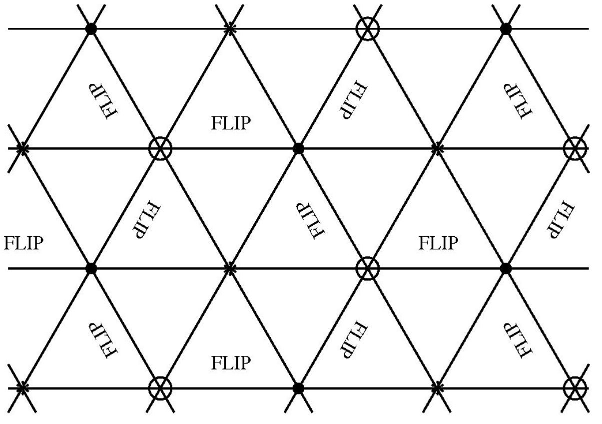


Figure 4: The $\{3,3,3\}$-tiling, with one tile and its images under $\widetilde{G}_3$ labeled.

illustrated in Figure 5. All of the vertices in this figure represent critical points for the projection map $\theta : \mathbb{C} \to \widehat{\mathbb{C}}$. Again lattice points have been circled. Any two adjacent small parallelograms form a fundamental region for the action of the group $\widetilde{G}_2$, which consists of 180° rotations around the vertices, together with lattice translations. The four small parallelograms adjacent to any vertex forms a fundamental domain under lattice translations.

In most cases, the corresponding flat orbifold is isometric to some tetrahedron in Euclidean space. (Compare [De].) For example, consider the case where the invariant γ

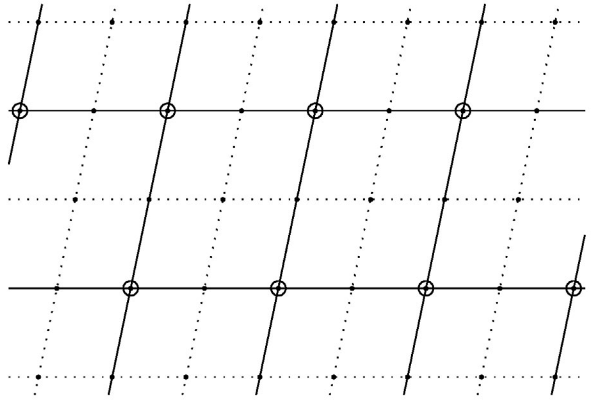


Figure 5: A typical $\{2,2,2,2\}$ tiling of the plane.

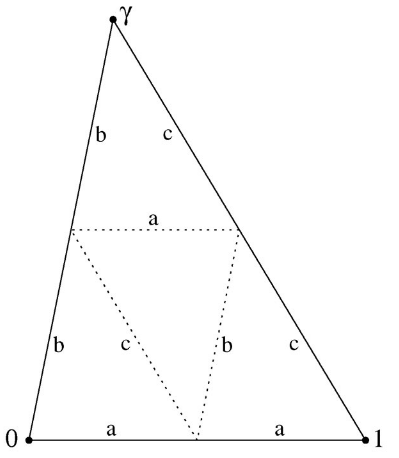


Figure 6: Illustrating the orbifold structure of $\mathcal{T}/G_2$.

in the Siegel domain (8) satisfies $0 < \Re(\gamma) < 1/2$. Then the triangle with vertices 0, 1 and γ has all angles acute, and also serves as a fundamental domain for the action of the crystallographic group $\widetilde{G}_2$ on $\mathbb{C}$. Joining the midpoints of the edges of this triangle, as shown in Figure 6, we can cut this triangle up into four similar triangles. Now fold along the dotted lines and bring the three corner triangles up so that the three vertices 0, 1 and γ come together. In this way, we obtain a tetrahedron which is isometric to the required flat orbifold $\mathcal{T}/G_2$. The construction when $-1/2 \le \Re(\gamma) < 0$ is the same, except that we use -1 and 0 in place of 0 and 1. The tetrahedrons which can be obtained in this way are characterized by the property that opposite edges have equal length, or by the property that there is a Euclidean motion carrying any vertex to any other vertex. In most cases this Euclidean motion is uniquely determined; but in the special case where we start with an equilateral triangle, with $\gamma = \omega_6$, we obtain a regular tetrahedron which has extra symmetries.

In the case $\Re(\gamma) = 0$ where γ is pure imaginary, this tetrahedron degenerates and the orbifold can be described rather as the "double" of the rectangle which has vertices

$$0,\ 1/2,\ \gamma/2,\ (1+\gamma)/2.$$

Again, in most cases there is a unique orientation preserving isometry carrying any vertex to any other vertex; but in the special case of a square, with $\gamma = i$, there are extra symmetries.

§5. Classification. By taking a closer look at the arguments in sections 3 and 4, we can give a precise classification of Lattès Maps. (Compare [DH, §9]; and see also [H, pp. 101–103] and [Mc3, p. 185].) It will be convenient to introduce the notation

$$\omega_n = \exp(2\pi i/n), \tag{12}$$

for the standard generator of the cyclic group G_n. Thus

$$\omega_2 = -1, \quad \omega_3 = (-1+i\sqrt{3})/2, \quad \omega_4 = i, \quad \omega_6 = \omega_3 + 1.$$

As usual, we consider a Lattès map which is conjugate to $L/G_n : \mathcal{T}/G_n \to \mathcal{T}/G_n$, where $\mathcal{T} \cong \mathbb{Z}/\Lambda$ and where $L(t) = at + b$. Here it will be convenient to think of b as a complex number, well defined modulo Λ.

Lemma 5.1. *Such a Lattès* map f *is uniquely determined up to conformal conjugacy by the following four invariants.*

- *First: the integer n, equal to 2, 3, 4, or 6.*
- *Second: the complex number a^n, with $|a|^2$ equal to the degree of f.*
- *Third: the lattice Λ, which we may take to have the form $\Lambda = \mathbb{Z} \oplus \gamma\mathbb{Z}$ with γ in the Siegel region* (8). *This lattice must satisfy the conditions that $\omega_n\Lambda = \Lambda$ and $a\Lambda \subset \Lambda$. Let k be the largest integer such that $\omega_k\Lambda = \Lambda$.*
- *Fourth: the product $(1-\omega_n)\, b \in \Lambda$ modulo the sublattice*

$$(1-\omega_n)\Lambda \; + \; (a-1)\Lambda \; \subset \; \Lambda,$$

up to multiplication by G_k with k as above. This last invariant is zero if and only if the map f admits a fixed point of maximal ramification index $r = n$, or equivalently a fixed point of multiplier $\mu = a^n$.

For most lattices we have $k = 2$, so that the image of $(1-\omega_n)\, b$ in the quotient group

$$\Lambda/((1-\omega_n)\Lambda \; + \; (a-1)\Lambda) \tag{13}$$

is invariant up to sign. However, in the special case where Λ has G_4 or G_6 symmetry, so that $\gamma = \omega_4$ or $\gamma = \omega_6$, this image is invariant only up to multiplication by ω_4 or ω_6 respectively.

Note that the first invariant n, equal to the greatest common divisor of the ramification indices, can easily be computed by looking at the orbits of the critical points of f, using formula (7) of §4. The invariant a^n can be computed from the multiplier μ at any fixed point, since the equation $\mu = (\omega a)^r$ of Corollary 3.9, with $\omega^n = 1$, implies that $\mu^{n/r} = a^n$. It follows from this equation that $\mu = a^n$ if and only if $r = n$.

The cases with $n \geq 3$ are somewhat easier than the case $n = 2$. In fact a normalized lattice Λ with G_3 or G_6 symmetry is necessarily equal to $\mathbb{Z}[\omega_3] = \mathbb{Z}[\omega_6]$, and the condition $a\Lambda \subset \Lambda$ is satisfied if and only if $a \in \mathbb{Z}[\omega_6]$. Similarly, G_4 symmetry implies that $\Lambda = \mathbb{Z}[i]$, and the possible choices for a are just the elements of $\mathbb{Z}[i]$, always subject to the standing requirement that $|a| > 1$.

Theorem 5.2. *If $n \geq 3$, then the conformal conjugacy class of f is completely determined by the numbers n and a^n where $a \in \mathbb{Z}[\omega_n]$, together with the information as to whether f does or does not have a fixed point of multiplier $\mu = a^n$.*

Evidently there is such a fixed point if and only if $(1-\omega_n)\,b$ is congruent to zero modulo $(1-\omega_n)\Lambda + (a-1)\Lambda$. (When $n=6$ there is necessarily such a fixed point.)

The proof of 5.1 and 5.2 will be based on the following.

Lemma 5.3. *The additive subgroup of $\mathcal{T} = \mathbb{C}/\Lambda$ consisting of elements fixed by the action of G_n is canonically isomorphic to the quotient group $\Lambda/(1-\omega_n)\Lambda$, of order $|1-\omega_n|^2 = 4\sin^2(\pi/n)$. In fact, the correspondence $t \mapsto (1-\omega_n)\,t$ maps the group of torus elements fixed by G_n isomorphically onto this quotient group.*

Proof. The required equation $\omega_n t \equiv t \mod \Lambda$ is equivalent to $(1-\omega_n)\,t \in \Lambda$, and the conclusion follows easily. □

Note that points of $\mathcal{T}$ fixed by the action of G_n correspond to points in the quotient sphere $\mathcal{T}/G_n$ of maximal ramification index $r=n$. As a check, in the four cases $\{2,2,2,2\}$, $\{3,3,3\}$, $\{2,4,4\}$, and $\{2,3,6\}$, there are clearly 4, 3, 2, and 1 such points respectively. This number is equal to $4\sin^2(\pi/n)$ in each case.

Proof of 5.1. It is clear that the numbers n, γ, a^n, and b completely determine the map $L/G_n : \mathcal{T}/G_n \to \mathcal{T}/G_n$. In fact γ determines the torus $\mathcal{T}$, and the power a^n determines a up to multiplication by n-th roots of unity. But we can multiply L and hence a by any n-th root of unity without changing the quotient L/G_n. Since the numbers n, a^n, and γ are uniquely determined by f (compare the discussion above), we need only study the extent to which b is determined by f.

Recall from Theorem 3.1 that the map $L(t) = at + b$ must commute with multiplication by ω_n. That is

$$L(\omega_n t) \equiv \omega_n L(t) \qquad \mod \Lambda.$$

Taking $t=0$ it follows that $b \equiv \omega_n b \mod \Lambda$, or in other words $(1-\omega_n)\,b \in \Lambda$, as required.

Next recall that we are free to choose any fixed point t_0 for the action of G_n on $\mathcal{T}$ as base point. Changing the base point to t_0 in place of 0 amounts to replacing $L(t)$ by the conjugate map $L(t+t_0) - t_0 = at + b'$ where

$$b' = b + (a-1)\,t_0, \quad \text{and hence} \quad (1-\omega_n)\,b' = (1-\omega_n)\,b + (a-1)(1-\omega_n)\,t_0.$$

Since the product $(1-\omega_n)\,t_0$ can be a completely arbitrary element of Λ, this means that we can add a completely arbitrary element of $(a-1)\Lambda$ to the product $(1-\omega_n)\,b$ by a change of base point. Thus the residue class

$$(1-\omega_n)\,b \in \Lambda/((1-\omega_n)\Lambda + (a-1)\Lambda),$$

together with n, γ, and a^n, suffices to determine the conjugacy class of f. However, we have not yet shown that this residue class is an invariant of f, since we must also consider automorphisms of $\mathcal{T}$ which fix the base point. Let ω be any root of unity which satisfies $\omega\Lambda = \Lambda$. Then $L(t) = at + b$ is conjugate to the map $\omega L(t/\omega) = at + \omega b$. In most cases, we can only choose $\omega = \pm 1$. (The fact that we are free to change the sign of b is irrelevant when n is even, but will be important in the case $n = 3$.) However, if $\Lambda = \mathbb{Z}[\omega_6]$ then we can choose ω to be any power of ω_6, and if $\Lambda = \mathbb{Z}[i]$ then we can choose ω to be any power of i. Further details of the proof are straightforward. □

Proof of Theorem 5.2. In the cases $n \geq 3$, we have noted that Λ is necessarily equal to $\mathbb{Z}[\omega_n]$. Furthermore, for $n = 3, 4, 6$, the quotient group $\Lambda/(1 - \omega_n)\Lambda$ is cyclic of order 3, 2, 1 respectively. Thus this group has at most one non-zero element, up to sign. The conclusion follows easily. □

The discussion of Lattès maps of type $\{2, 2, 2, 2\}$ will be divided into two cases according as $a \in \mathbb{Z}$ or $a \notin \mathbb{Z}$. First suppose that $a \notin \mathbb{Z}$.

Definition. A complex number a will be called an *imaginary quadratic integer* if it satisfies an equation $a^2 - qa + d = 0$ with integer coefficients and with $q^2 < 4d$, so that

$$a = (q \pm \sqrt{q^2 - 4d})/2 \tag{14}$$

is not a real number. Here $|a|^2 = d$ is the associated degree. Evidently the imaginary quadratic integers form a discrete subset of the complex plane. In fact for each choice of $|a|^2 = d$ there are roughly $4\sqrt{d}$ possible choices for q, and twice that number for a.

Lemma 5.4. *A complex number a can occur as the derivative $a = L'$ associated with an affine torus map if and only if it is either a rational integer $a \in \mathbb{Z}$ or an imaginary quadratic integer. If $a \in \mathbb{Z}$ then any torus can occur, but if $a \notin \mathbb{Z}$ then there are only finitely many possible tori up to conformal diffeomorphism. Furthermore, there is a one-to-one correspondence between conformal diffeomorphism classes of such tori and ideal classes in the ring $\mathbb{Z}[a]$.*

Proof. Let $\mathcal{T} = \mathbb{C}/\Lambda$. The condition that $a\Lambda \subset \Lambda$ means that Λ must be a module over the ring $\mathbb{Z}[a]$ generated by a. We first show that a must be an algebraic integer. Without loss of generality, we may assume that $\Lambda = \mathbb{Z} \oplus \gamma\mathbb{Z}$ is a normalized lattice, satisfying the Siegel conditions (8). Thus $1 \in \Lambda$ and it follows that all powers of a belong to Λ. If Λ_k is the sublattice spanned by $1,\ a,\ a^2, \ldots,\ a^k$, then the lattices $\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda$ cannot all be distinct. Hence some power a^k must belong to Λ_{k-1}, which proves that a satisfies a monic equation with integer coefficients, and hence is an algebraic integer. On the other hand, a belongs to a quadratic number field since the three numbers $1,\ a,\ a^2 \in \Lambda$ must satisfy a linear relation with integer coefficients. Using the fact that the ring of integer polynomials forms a unique factorization domain, it follows that a satisfies a monic degree two polynomial.

Now given $a \notin \mathbb{Z}$ we must ask which normalized lattices Λ are possible. Since $a \in \Lambda$, we can write $a = r + s\gamma$ with $r,\ s \in \mathbb{Z}$. Changing the sign of a if necessary, we may assume that $s > 0$. Taking real and imaginary parts, it follows that

$$r = \Re(a) - s\,\Re(\gamma) \qquad \text{and} \qquad s = \Im(a)/\Im(\gamma).$$

On the other hand, it follows easily from the Siegel inequalities

$$|\gamma| \geq 1, \qquad |\Re(\gamma)| \leq 1/2, \qquad \Im(\gamma) > 0$$

that $\Im(\gamma) \geq \sqrt{3}/2$. Since a has been specified, this inequality yields an upper bound of $2\,\Im(a)/\sqrt{3}$ for s, and the inequality $|\Re(\gamma)| \leq 1/2$ then yields an upper bound for $|r|$. Thus there are only finitely many possibilities for $\gamma = (a - r)/s$.

Next note that the product lattice $\mathcal{I} = s\,\Lambda = s\mathbb{Z} \oplus (a-r)\mathbb{Z}$ is contained in the ring $\mathbb{Z}[a]$, and is an ideal in this ring since $a\,\mathcal{I} \subset \mathcal{I}$. Clearly the torus $\mathbb{C}/\Lambda$ is isomorphic to $\mathbb{C}/\mathcal{I}$. If $\mathcal{I}'$ is another ideal in $\mathbb{Z}[a]$, note that $\mathbb{C}/\mathcal{I} \cong \mathbb{C}/\mathcal{I}'$ if and only if $\mathcal{I}' = c\,\mathcal{I}$ for some constant $c \neq 0$. Such a constant must belong to the quotient field $\mathbb{Q}[a]$, so by definition this means that $\mathcal{I}$ and $\mathcal{I}'$ represent the same ideal class. □

For further discussion of maps with imaginary quadratic a see 7.2, 8.1 and 8.2 below. We next discuss the case $a \in \mathbb{Z}$.

Definition. A Lattès map

$$L/G_n : \mathcal{T}/G_n \to \mathcal{T}/G_n \qquad \text{with} \qquad \mathcal{T} = \mathbb{C}/\Lambda$$

will be called *flexible* if we can vary Λ and L continuously so as to obtain other Lattès maps which are not conformally conjugate to it.

Lemma 5.5. *A Lattès* map $L/G_n : \mathcal{T}/G_n \to \mathcal{T}/G_n$ *is flexible if and only if* $n = 2$ *and the affine* map $L(\tau) = a\tau + b$ *has integer derivative,* $L' = a \in \mathbb{Z}$.

Proof. This follows easily from 5.1 and 5.4. □

We can easily classify such maps into a single connected family provided that the degree a^2 is even, and into two connected families when a^2 is odd, as follows. In each case, the coefficients a and b will remain constant but $\mathcal{T}$ will vary through all possible conformal diffeomorphism classes.

- **Maps with postcritical fixed point.** Let $\mathbb{H}$ be the upper half-plane. For each integer $a \geq 2$ there is a connected family of flexible Lattès maps of degree a^2 parametrized by the half-cylinder $\mathbb{H}/\mathbb{Z}$, as follows. Let $\mathcal{T}(\gamma)$ be the torus $\mathbb{C}/(\mathbb{Z} \oplus \gamma\mathbb{Z})$ where γ varies over $\mathbb{H}/\mathbb{Z}$, and let $L : \mathcal{T}(\gamma) \to \mathcal{T}(\gamma)$ be the map $L(t) = at$. Then

$$L/G_2 : \mathcal{T}(\gamma)/G_2 \to \mathcal{T}(\gamma)/G_2$$

is the required smooth family of maps, with the image of $0 \in \mathcal{T}$ as ramified fixed point. If we restrict γ to the Siegel region (8), then we get a set of representative maps which are unique up to holomorphic conjugacy.

• **Maps without postcritical fixed point.** The construction in this case is identical, except that we take $L(t) = at + 1/2$. If a is even, this construction yields nothing new. In fact, the quotient group (13) of 5.1 is then trivial, and it follows that every Lattès map with $L' = a$ must have a postcritical fixed point. However, when a is odd, the period two orbits

$$0 \leftrightarrow 1/2, \qquad \gamma/2 \leftrightarrow (\gamma+1)/2$$

in $\mathcal{T}(\gamma)$ map to ramified period two orbits in $\mathcal{T}(\gamma)/G_2$, and there is no postcritical fixed point.

Caution: In this last case, we can no longer realize every conjugacy class of maps by restricting γ to the Siegel region. A larger fundamental domain is needed. For explicitly worked out examples in both cases, see equations (15), (18) and (19) below; and for further discussion see §7.

Here is another characterization.

Lemma 5.6. *A Lattès map is flexible if and only if the multiplier for every periodic orbit is an integer.*

Proof. If $n = 2$ and $a \in \mathbb{Z}$, then it follows immediately from Corollary 3.9 that every multiplier is an integer. On the other hand, if $n > 2$ or if $a \notin \mathbb{Z}$, then we can find infinitely many integers $p > 0$ so that $\omega\, a^p \notin \mathbb{Z}$ for some $\omega \in G_n$. The number of fixed points of the map $\omega\, L^{\circ p}$ on the torus $\mathcal{T}$ grows exponentially with p (the precise number is $|\omega\, a^p - 1|^2$), and each of these maps to a periodic point of the associated Lattès map f. If we exclude the three or four postcritical points, then the derivative of $f^{\circ p}$ at such a point will be $\omega\, a^p$, so that the multiplier of this periodic orbit cannot be an integer. □

It seems very likely that power maps, Chebyshev maps, and flexible Lattès maps are the only rational maps such that the multiplier of every periodic orbit is an integer. (For a related result, see Lemma 7.1 below.)

§6. Lattès Maps before Lattès. Although the name of Lattès has become firmly attached to the construction studied in this paper, it actually occurs much earlier in the mathematical literature. Ernst Schröder [S], in a well known paper written in 1871, first used trigonometric identities to construct examples of what I call "Chebyshev maps". He then constructed an explicit one-parameter family of "Lattès" type examples as follows. Let $x = \mathrm{sn}(u)$ be the Jacobi sine function with elliptic modulus k, defined by the equation

$$u = \int_0^x \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}.$$

More explicitly, for any parameter $k^2 \neq 0, 1$, let $E_k \subset \mathbb{C}^2$ be the elliptic curve defined by the equation $y^2 = (1 - x^2)(1 - k^2x^2)$. Then the holomorphic differential dx/y is smooth and non-zero everywhere on E_k (even at the two points at infinity in terms of suitable local coordinates). The integrals $\oint dx/y$ around cycles in E_k generate a lattice $\Lambda \subset \mathbb{C}$, and the integral

$$u(x, y) = \int_{(0,1)}^{(x,y)} dx/y$$

along any path from $(0, 1)$ to (x, y) in E_k is well defined modulo this lattice. In fact the resulting coordinate u parametrizes the torus $\mathcal{T} = \mathbb{C}/\Lambda$, and we can set $x = \mathrm{sn}(u)$ and $y = \mathrm{cn}(u)\,\mathrm{dn}(u)$. Here $\mathrm{sn}(u)$ is the Jacobi sine function, and $\mathrm{cn}(u)$ and $\mathrm{dn}(u)$ are closely related doubly periodic meromorphic functions which satisfy

$$\mathrm{cn}^2(u) = 1 - \mathrm{sn}^2(u) \qquad \text{and} \qquad \mathrm{dn}^2(u) = 1 - k^2\mathrm{sn}^2(u).$$

Furthermore

$$\mathrm{sn}(2u) = \frac{2\,\mathrm{sn}(u)\,\mathrm{cn}(u)\,\mathrm{dn}(u)}{1 - k^2\mathrm{sn}^4(u)}.$$

(Compare [WW, §22.2].) Setting $z = x^2 = \mathrm{sn}^2(u)$ it follows easily that there is a well defined rational function

$$f(z) = \frac{4z(1-z)(1-k^2z)}{(1-k^2z^2)^2} \tag{15}$$

of degree four which satisfies the semiconjugacy relation

$$\mathrm{sn}^2(2u) = f(\mathrm{sn}^2(u)).$$

This is Schröder's example (modulo a minor misprint). In the terminology of §5, f is a "flexible Lattès map", described 47 years before Lattès.

It is not hard to see that $\mathrm{sn}(u)$ has critical values ± 1 and $\pm 1/k$, and hence that $\mathrm{sn}^2(u)$ has critical values 1, $1/k^2$, 0, and ∞. On the other hand the map f has three critical values 1, $1/k^2$ and ∞, which all map to the fixed point $0 = f(0)$. Each of these three critical values is the image under f of two distinct simple critical points.

Lucyan Böttcher cited the same example in 1904, with a different version of the misprint. (See [Bö].) He was perhaps the first to think of this example from a dynamical viewpoint, and to use the term "chaotic" to describe the behavior of the sequence of iterates of f. In fact he described an orbit $z_0 \mapsto z_1 \mapsto \cdots$ as *chaotic* if for every convergent subsequence $\{z_{n_i}\}$ the differences $n_{i+1} - n_i$ are unbounded. (Note that this includes examples such as irrational rotations which are not chaotic in the modern sense.)

Böttcher actually cited a much earlier paper, written by Charles Babbage in 1815, for a fundamental property of what we now call semiconjugacies. For example, in order to find a periodic point $\psi^n x = x$ of a mapping ψ, Babbage proceeded as follows (see [Ba, p. 412]):

"*Assume as before* $\psi x = \phi^{-1} f \phi\, x$*, then*

$$\psi^2 x = \phi^{-1} f \phi \phi^{-1} f \phi\, x = \phi^{-1} f^2 \phi\, x$$
$$\psi^3 x = \phi^{-1} f^2 \phi \phi^{-1} f \phi\, x = \phi^{-1} f^3 \phi\, x,$$

and generally $\psi^n x = \phi^{-1} f^n \phi\, x$*, hence our equation becomes*

$$\phi^{-1} f^n \phi\, x = x. \ldots"$$

In modern terminology, we would say that ϕ is a *semiconjugacy* from ψ to f. It follows that any periodic point of ψ maps to a periodic point of f; and furthermore (assuming that ϕ is finite-to-one and onto) any periodic point of f is the image of a periodic point of ψ. Böttcher pointed out that the use of such an intermediary map ϕ to relate the dynamic properties of two maps ψ and f lies at the heart of Schröder's example.

J. F. Ritt carried out many further developments of these ideas in the 1920's. (Compare [R1], [R2], [R3], and see the "Ritt-Eremenko Theorem" in §7. For further historical information, see [A].)

§7. More Recent Developments. This concluding section will outline some of the special properties shared by some or all finite quotients of affine maps.

We first consider the class of *flexible* Lattès maps, as described in 5.5 and 5.6. These are the only known rational maps without attracting cycles which admit a continuous family of deformations preserving the topological conjugacy class. In fact the C^∞ conjugacy class remains almost unchanged as we deform the torus. Differentiability fails only at the postcritical points; and the multipliers of periodic orbits remain unchanged even at these postcritical points.

Closely related is the following:

Fundamental Conjecture. The flexible Lattès maps are the only rational maps which admit an "invariant line field" on their Julia set.

By definition f has an *invariant line field* if its Julia set J has positive Lebesgue measure, and if there is a measurable f-invariant field of real one-dimensional subspaces of the tangent bundle of $\widehat{\mathbb{C}}$ restricted to J. The importance of this conjecture is demonstrated by the following. (See [MSS], and compare the discussion in [Mc2] as well as [BM].)

Theorem of Mañé, Sad and Sullivan. *If this Fundamental Conjecture is true, then hyperbolicity is dense among rational maps. That is, every rational map can be approximated by a hyperbolic map.*

To see that every flexible Lattès map has such an invariant line field, note that any torus $\mathbb{C}/(\mathbb{Z} \oplus \gamma\mathbb{Z})$ is foliated by a family of circles $\Im(t) =$ constant which is invariant under the affine map L. If f is the associated Lattès map L/G_2, then this circle foliation maps to an f-invariant foliation of $J(f) = \widehat{\mathbb{C}}$ which is not only measurable but actually smooth, except for singularities at the four postcritical points.

Let us define the *multiplier spectrum* of a degree d rational map f to be the function which assigns to each $p \geq 1$ the unordered list of multipliers at the $d^p + 1$ (not necessarily distinct) fixed points of the iterate $f^{\circ p}$. Call two maps *isospectral* if they have the same multiplier spectrum.

Theorem of McMullen. *The flexible Lattès maps are the only rational maps which admit non-trivial isospectral deformations. The conjugacy class of any rational map which is not flexible Lattès is determined, up to finitely many choices, by its multiplier spectrum.*

This is proved in [Mc1, §2]. I am grateful to McMullen for the observation that the Lattès maps L/G_2 associated with imaginary quadratic number fields provide a rich source of isospectral examples which are not flexible. First note the following.

Lemma 7.1. *Two Lattès maps $L/G_2 : \mathcal{T}/G_2 \to \mathcal{T}/G_2$ are isospectral if and only if they have the same derivative $L' = a$, up to sign, and the same numbers of periodic orbits of various periods within the postcritical set P_f.*

Proof Outline. Let $\omega = \pm 1$. The number of fixed points of the map $\omega L^{\circ p}$ on the torus can be computed as $|\omega a^p - 1|^2$. These fixed points occur in pairs $\{\pm t\}$, and each such pair corresponds to a single fixed point of the corresponding iterate $f^{\circ p}$, where $f \cong L/G_2$. Whenever $t \neq -t$ on the torus, the multiplier of $f^{\circ p}$ at this fixed point is also equal to ωa^p. For the exceptional points $t = -t$ which are fixed under the action of G_2 and correspond to postcritical points of f, the multiplier of $f^{\circ p}$ is equal to a^{2p}. Thus, to determine the multiplier spectrum completely, we need only know how many points of various periods there are in the postcritical set. □

Example 7.2. Let $\xi = i\sqrt{k}$ where $k > 0$ is a square-free integer, and let $a = m\xi + n$. Then for each divisor d of m the lattice $\mathbb{Z}[\xi d] \subset \mathbb{C}$ is a $\mathbb{Z}[a]$-module. Hence the linear map $L(\tau) = a\tau$ acts on the associated torus $\mathcal{T}/\mathbb{Z}[\xi d]$. If m is highly divisible, then there are many possible choices for d. Suppose, to simplify the discussion, that mk and n are both even, so that a^2 is divisible by two in $\mathbb{Z}[a]$. Then multiplication by a^2 acts as the zero map on the group consisting of elements of order two in $\mathcal{T}$. Thus $0 = L(0)$ is the only periodic point under this action, hence the image of 0 in $\mathcal{T}/G_2$ is the only postcritical periodic point of L/G_2. It then follows from 7.1 that these examples are all isospectral.

Berteloot and Loeb [BL1] have characterized Lattès maps in terms of the dynamics and geometry of the associated homogeneous polynomial map of $\mathbb{C}^2$. Every rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree two or more lifts to a homogeneous polynomial map $F : \mathbb{C}^2 \to \mathbb{C}^2$ of the same degree with the origin as an attracting fixed point. They show that f is Lattès if and only if the boundary of the basin of the origin under F is smooth and strictly pseudoconvex within some open set. In fact, this boundary must be "spherical", except over the postcritical set of f; that is, it can be transformed to the standard sphere

$\sum |z_j|^2 = 1$ by a local holomorphic change of coordinates. For the corresponding result in higher dimensions, see [Du]. (For higher dimensional Lattès maps, see 7.3.)

Anna Zdunik [Z] has characterized Lattès maps using only measure theoretic properties. It is not hard to see that the standard probability measure on the flat torus pushes forward under Θ to an ergodic f-invariant probability measure on the Riemann sphere. This measure is smooth and in fact real analytic, except for mild singularities at the postcritical points. Furthermore, it is *balanced*, in the sense that the preimage $f^{-1}(U)$ of any simply connected subset of $\widehat{\mathbb{C}} \setminus P_f$ is a union of n disjoint sets of equal measure. Hence it coincides with the unique measure of maximal entropy, as constructed by Lyubich [Ly], or by Freire, Lopez and Mañé [FLM], [Ma]. The converse theorem is much more difficult:

Theorem of Zdunik. *The Lattès maps are the only rational maps for which the measure of maximal entropy is absolutely continuous with respect to Lebesgue measure.*

The same result for higher dimensional maps has been proved by Berteloot and Dupont [BD]. They also characterize higher dimensional Lattès maps by minimality of the Lyapunov exponents or maximality for the Hausdorff dimension associated with their measure of maximal entropy.

We can think of the maximal entropy measure $\mu_{\max}$ as describing the asymptotic distribution of random backward orbits. That is, if we start with any non-exceptional initial point z_0, and then use a fair d-sided coin or die to iteratively choose a backward orbit

$$\cdots \mapsto z_{-2} \mapsto z_{-1} \mapsto z_0,$$

then $\{z_n\}$ will be equidistributed with respect to $\mu_{\max}$. This measure $\mu_{\max}$ always exists. An absolutely continuous invariant measure is much harder to find, and an invariant measure which is ergodic and belongs to the same measure class as Lebesgue measure is even rarer. However Lattès maps are not the only examples: Mary Rees [Re] has shown that for every degree $d \geq 2$ the moduli space of degree d rational maps has a subset of positive measure consisting of maps f which have an ergodic invariant measure μ in the same measure class as Lebesgue measure. Such a measure is clearly unique, since Lebesgue almost every forward orbit $z_0 \mapsto z_1 \mapsto z_2 \cdots$ must be equidistributed with respect to μ.

Using these ideas, an easy consequence of Zdunik's Theorem is the following.

Corollary. *A Lebesgue randomly chosen forward orbit for a Lattès map has the same asymptotic distribution as a randomly chosen backward orbit.*

I don't know whether Lattès maps are the only maps of degree $d \geq 2$ with this property.

In general, different rational maps have different invariant measures, except that every invariant measure for f is also an invariant measure for its iterates $f^{\circ p}$. However, every Lattès map L/G_n shares its measure $\mu_{\max}$ with a rich collection of Lattès maps

$\widehat{L}/G_n$ where $\widehat{L}$ ranges over all affine maps of the torus which commute with the action of G_n. This collection forms a semigroup which is not finitely generated. (If we consider only the linear torus maps $\widehat{L}(\tau) = a\tau$, then we obtain a commutative semigroup.) I don't know any other examples, outside of the Chebyshev and power maps, of a semigroup of rational maps which is not finitely generated, and which shares a common non-atomic invariant measure. (See [LP] for related results.)

Closely related is the study of commuting rational maps. Following a terminology introduced much later by Veselov [V], let us call a rational map f *integrable* if it commutes with another rational map, $f \circ g = g \circ f$, where both f and g have degree at least two, and where no iterate of f is equal to an iterate of g.

Theorem of Ritt and Eremenko. *A rational map f of degree $d_f \geq 2$ is integrable if and only if it is a finite quotient of an affine map; that is if and only if it is either a Lattès, Chebyshev, or power map. Furthermore, the commuting map g must have the same Julia set, the same flat orbifold metric, the same measure of maximal entropy, and the same set of preperiodic points as f.*

This is a modern formulation of a statement which was proved by Ritt [R2] in 1923, and by Eremenko [E] using a quite different method in 1989. For higher dimensional analogues, see [Di2], [DS].

Remark 7.3. There has been a great deal of interest in higher dimensional analogues of Lattès maps in recent years. Let $\widetilde{G}$ be a complex crystallographic group of automorphisms of $\mathbb{C}^r$, that is a discrete group of complex affine maps with compact quotient. (Such maps were characterized by Bieberbach, also in the real case—see for example [M2].) If $L : \mathbb{C}^r \to \mathbb{C}^r$ is an affine map, and if $L \circ g \circ L^{-1} \in \widetilde{G}$ for every $g \in \widetilde{G}$, then L induces a holomorphic self-map of the quotient orbifold $\mathbb{C}^r/\widetilde{G}$. We are particularly interested in the case where this quotient can be identified with complex projective r-space. As an example, given any one-dimensional example with crystallographic group $\widetilde{G}_1$ and affine map L_1, let $\widetilde{G}$ be generated by the r-fold cartesian product $\widetilde{G}_1 \times \cdots \times \widetilde{G}_1$, together with the symmetric group of permutations of the r coordinates. Then the quotient $\mathbb{C}^r/\widetilde{G}$ can indeed be identified with complex projective space, and the map $L = L_1 \times \cdots \times L_1$ of $\mathbb{C}^r$ gives rise to a "Lattès map" of this quotient r-dimensional projective space.

For further information about higher dimensional Lattès maps, compare [BL2], [Di1], [Di2], [Du], and [V].

§8. Examples. This concluding section will provide explicit formulas for some particular Lattès maps.

8.1. Degree Two Lattès Maps. Recall from Lemma 5.4 and equation (14) that the derivative $L' = a$ for a torus map of degree d must either be a (rational) integer, so that

$d = a^2$, or must be an imaginary quadratic integer of the form $a = (q \pm \sqrt{q^2 - 4d})/2$ with $q^2 < 4d$, satisfying $a^2 - q\,a + d = 0$ and $|a|^2 = d$. Furthermore, replacing a by $-a$ if necessary, we may assume that $q \geq 0$. Thus, in the degree two case, the only distinct possibilities are $q = 0, 1, 2$, with

$$a = i\sqrt{2}, \qquad \text{or} \qquad a = (1 \pm i\sqrt{7})/2, \qquad \text{or} \qquad a = 1 \pm i.$$

In each of these cases, the associated torus $\mathcal{T} = \mathbb{C}/\Lambda$ is necessarily conformally isomorphic to the quotient $\mathbb{C}/\mathbb{Z}[a]$. In fact we can assume that $\Lambda = \mathbb{Z} \oplus \gamma\mathbb{Z}$ with γ in the Siegel region (8), and set $a = r + s\gamma$ with $r,\ s \in \mathbb{Z}$. Let us assume, to fix our ideas, that $\Im(a) > 0$. Then:

$$0 < s = \Im(a)/\Im(\gamma) \leq \frac{2\,\Im(a)}{\sqrt{3}} \leq \frac{2|a|}{\sqrt{3}} = \frac{2\sqrt{2}}{\sqrt{3}} \approx 1.63.$$

Therefore $s = 1$, hence $a \equiv \gamma (\mathrm{mod}\mathbb{Z})$; so the lattice Λ must be equal to $\mathbb{Z}[a]$.

First suppose that $f \cong L/G_2$ is a Lattès map of type $\{2,2,2,2\}$, with $L(t) = at + b$. The four points of the form $\lambda/2$ in $\mathcal{T}$ map to the four postcritical points of f. Hence the action of the Lattès map f on its postcritical set is mimicked by the action of L on this group of elements of the form $\lambda/2$ in $\mathcal{T}$, or equivalently by the action of $t \mapsto at + 2b$ on the four element group $\mathbb{Z}[a]/2\mathbb{Z}[a]$. A brief computation shows that the quotient group $\mathbb{Z}[a]/(2\mathbb{Z}[a] + (a-1)\mathbb{Z}[a])$ of Lemma 5.1 is trivial when q is even but has two elements when q is odd. Thus, in the two cases $a = i\sqrt{2}$ and $a = 1 + i$ where q is even, we may assume that $L(t) = at$. In these cases, the equation $a^2 - qa + 2 = 0$ implies that $a^2 \equiv 0 \mod 2\mathbb{Z}[a]$, and hence that

$$1 \mapsto a \mapsto 0 \quad \text{and} \quad 1 + a \mapsto a \mapsto 0 \quad (\text{modulo } 2\mathbb{Z}[a])$$

under multiplication by a. Thus in these two cases there is a unique postcritical fixed point, represented by 0, where the multiplier at this fixed point must be a^2 by 3.9. In fact, the diagram of critical and postcritical points for the Lattès map f necessarily has the following form.

$$\begin{array}{ccccccccc} & & & & \circledcirc & & & & \\ & & & & \uparrow & & & & \\ * & \mapsto & \bullet & \mapsto & \bullet & \leftarrow\!\shortmid & \bullet & \leftarrow\!\shortmid & * \end{array}$$

Here each star stands for a simple critical point, each heavy dot stands for a ramified (or postcritical) point, and the heavy dot with a circle around it stands for a postcritical fixed point. If we put the two critical points at ± 1 and put the postcritical fixed point at infinity, then f will have the form

$$f(z) = (z + z^{-1})/a^2 + c$$

for some constant c. (Compare [M3].) In fact it is not difficult to derive the forms

$$\begin{aligned} f(z) &= -(z+z^{-1})/2+\sqrt{2} \quad \text{when} \quad a = i\sqrt{2}, \quad \text{and} \\ f(z) &= \pm(z+z^{-1})/2i \qquad \text{when} \quad a = 1 \pm i. \end{aligned}$$

On the other hand, for $a = (1 \pm i\sqrt{7})/2$, a similar argument shows that there are two possible critical orbit diagrams, as follows. Either:

$$\mapsto \quad \bullet \quad \mapsto \quad \odot \qquad\qquad * \quad \mapsto \quad \bullet \quad \mapsto \quad \odot$$

with two postcritical fixed points, or

$$* \quad \mapsto \quad \bullet \quad \mapsto \quad \bullet \quad \longleftrightarrow \quad \bullet \quad \mapsfrom \quad \bullet \quad \mapsfrom \quad *$$

with no postcritical fixed points. In the first case, if we put the postcritical fixed points at zero and infinity, and another fixed point at $+1$, then the map takes the form

$$f(z) = z\frac{z+a^2}{a^2z+1}.$$

This commutes with the involution $z \mapsto 1/z$, and we can take the composition

$$z \mapsto f(1/z) = 1/f(z) = \frac{az^2+1}{z(z+a^2)}$$

as the other Lattès map with the same value of a, but with $\{0, \infty\}$ as postcritical period two orbit. (See [M6, §B.3] for further information on these maps.[6])

We can also ask for Lattès maps of degree two of the form L/G_n with $n > 2$. However, only the type $\{2, 4, 4\}$ with $n = 4$ can occur, since, of the lattices $\mathbb{Z}[a]$ described above, only $\mathbb{Z}[1+i] = \mathbb{Z}[i]$ admits a rotation of order greater than 2. In fact the rotation $t \mapsto it$ of the torus $\mathbb{C}/\mathbb{Z}[i]$ corresponds to an involution $z \mapsto -z$ which commutes with the associated Lattès map $f(z) = (z+z^{-1})/2i$. To identify z with $-z$, we can introduce the new variable $w = z^2$ and set

$$g(w) = g(z^2) = f(z)^2 = -(z^2+2+z^{-2})/4 = -(w+2+w^{-1})/4.$$

Up to holomorphic conjugacy, this is the unique degree two Lattès map of the form L/G_4. Its critical points ± 1 have orbit $1 \mapsto -1 \mapsto 0 \mapsto \infty \circlearrowleft$, so that -1 is both a critical point and a critical value, yielding the following schematic diagram.

$$\begin{array}{ccccccc} * & \mapsto & * & \mapsto & \bullet & \mapsto & \odot \\ & & 2 & & 4 & & 4 \end{array}$$

[6] Caution: In both [M4, 2000] and [M6], the term "Lattès map" was used with a more restricted meaning, allowing only maps of type $\{2, 2, 2, 2\}$ with $n = 2$.)

Here the ramification index is indicated underneath each ramified point. Thus the map has type $\{2,4,4\}$, as expected. (Alternatively, the map $z \mapsto 1 - 2/z^2$, with critical points zero and infinity and with critical orbit $0 \mapsto \infty \mapsto 1 \mapsto -1 \circlearrowleft$, could also be used as a normal form for this same conjugacy class.)

Remark. It is interesting to note that every quadratic Lattès map can be constructed as a mating $p_1 \perp\!\!\!\perp p_2$, where p_1 and p_2 are quadratic polynomial maps which have dendrites as Julia sets. (Compare [M6, §B.8]. In fact, in the special case $a = (1 \pm i\sqrt{7})/2$ with postcritical fixed points, there are four essentially distinct representations as a mating.) Each such mating structure gives rise to a continuous map $\mathbb{R}/\mathbb{Z} \to \widehat{\mathbb{C}}$ which semiconjugates the doubling map on the circle to the Lattès map. Hubbard has asked whether a corresponding statement holds for higher degree Lattès maps.

8.2. Degree Three. If the torus map $L(t) = at + b$ has degree $|a|^2 = 3$, then according to equation (14) we can write $a = (q \pm \sqrt{q^2 - 12})/2$ for some integer q with $q^2 < 12$ or in other words $|q| \le 3$. I will try to analyze only a single case, choosing $q = 0$ with $a = i\sqrt{3}$ so that $a^2 = -3$.

For this choice $a = i\sqrt{3}$, setting $a = r + s\gamma$ as in the proof of 5.4, we find that s can be either one or two, and it follows easily that there are exactly two essentially distinct tori which admit an affine map L with derivative $L' = a$. We can choose either the hexagonally symmetric torus $\mathcal{T} = \mathbb{C}/\mathbb{Z}[\omega_6] = \mathbb{C}/\mathbb{Z}[(a+1)/2]$, or its 2-fold covering torus $\mathcal{T}' = \mathbb{C}/\mathbb{Z}[a]$.

For the torus $\mathbb{C}/\mathbb{Z}[a]$, since there is no G_3 or G_4 symmetry, we are necessarily in the case $n = 2$. A brief computation shows that the quotient group $\mathbb{Z}[a]/(2\mathbb{Z}[a] + (a-1)\mathbb{Z}[a])$ of Lemma 5.1 has two elements, so there are two possible Lattès maps, corresponding to the two affine maps $L(t) = at$ and $L(t) = at + 1/2$. In the first case, the corresponding critical orbit diagram has the form

$$* \mapsto \odot \qquad * \mapsto \odot \qquad * \mapsto \bullet \leftrightarrow \bullet \mapsfrom *$$

with two postcritical fixed points. In the second case, it takes the form

$$\begin{array}{ccccccc} * & \mapsto & \bullet & \mapsto & \bullet & \mapsfrom & * \\ & & \uparrow & & \downarrow & & \\ * & \mapsto & \bullet & \mapsfrom & \bullet & \mapsfrom & * \end{array}$$

with a period four orbit of postcritical points. In the first case, if we place the postcritical fixed points at zero and infinity, and place a fixed point with multiplier $+a$ at $+1$, then the map takes the form

$$f(z) = \frac{z(z-a)^2}{(az-1)^2}.$$

The remaining fixed point then lies at -1 and has multiplier $-a$. The two remaining critical points $\pm 2i - a$ map to the period two postcritical orbit $-2i + a \longleftrightarrow 2i + a$.

We can construct the other Lattès maps with the same a and the same lattice $\mathbb{Z}[a]$ by composing f with the Möbius involution g which satisfies

$$g : 0 \leftrightarrow 2i + a, \qquad g : \infty \leftrightarrow -2i + a.$$

The critical orbit diagram for this composition permutes the four postcritical points cyclically, as required.

A beautifully symmetric example. Now consider the torus $\mathcal{T} = \mathbb{C}/\mathbb{Z}[\omega_6]$. As noted at the end of §4, the quotient $\mathcal{T}/G_2$, with its flat orbifold metric, is isometric to a regular tetrahedron with the four cone points as vertices. Again there are two distinct Lattès maps with invariant $a^2 = -3$, according as there is or is not a postcritical fixed point. The map $L(t) = at$ induces a highly symmetric piecewise linear map L/G_2 of this tetrahedron. (Compare [DMc] for a discussion of symmetric rational maps.) The four vertices are postcritical fixed points of this map, and the midpoints of the four faces are the critical points, each mapping to the opposite vertex. Thus the critical orbit diagram has the following form.

$$* \mapsto \odot \qquad * \mapsto \odot \qquad * \mapsto \odot \qquad * \mapsto \odot$$

The midpoint of each edge maps to the midpoint of the opposite edge, thus forming three period two orbits.

If we place these critical points on the Riemann sphere at the cube roots of -1 and at infinity, then this map takes the form

$$f(z) = \frac{6\,z}{z^3 - 2}, \tag{16}$$

with a critical orbit $\omega \mapsto -2\omega \circlearrowleft$ whenever $\omega^3 = -1$, and also $\infty \mapsto 0 \circlearrowleft$.

The affine map $L(t) = at + 1/2$ yields a Lattès map L/G_2 with the same critical and postcritical points, but with the following critical orbit diagram.

$$* \mapsto \bullet \leftrightarrow \bullet \leftarrow\!\dashv * \qquad * \mapsto \bullet \leftrightarrow \bullet \leftarrow\!\dashv *$$

Such a map can be constructed by composing the map f of (16) with the Möbius involution

$$g(z) = (2 - z)/(1 + z)$$

which satisfies $-1 \leftrightarrow \infty$ and $\omega_6 \leftrightarrow \overline{\omega}_6$. This corresponds to a 180° rotation of the tetrahedron about an axis joining the midpoints of two opposite faces.

Now consider Lattès maps L/G_n with $n \geq 3$ and with $a = i\sqrt{3}$. Evidently the lattice must be $\mathbb{Z}[\omega_6]$, and n must be either 3 or 6, so the type must be either $\{3, 3, 3\}$ or $\{2, 3, 6\}$. Using Theorem 5.2, we can easily check that there is just one possible map in each case, corresponding to the linear map $L(t) = at$. Since both G_2 and G_3

are subgroups of G_6, this torus map $L(t)=at$ gives rise to maps of type $\{2, 2, 2, 2\}$ and $\{3, 3, 3\}$ and $\{2, 3, 6\}$ which are related by the commutative diagram

$$\begin{array}{ccc} L & \mapsto & L/G_2 \\ \downarrow & & \downarrow \\ L/G_3 & \mapsto & L/G_6. \end{array}$$

Here L/G_2 is the "beautifully symmetric example" of equation (16). The corresponding Lattès map L/G_6 of type $\{2, 3, 6\}$ can be constructed from (16) by identifying each z with ωz for $\omega \in G_3$. If we introduce the new variable $\zeta = z^3$, then the corresponding map $L/G_6 = f/G_3$ is given by mapping $\zeta = z^3$ to $g(\zeta) = f(z)^3$, so that

$$g(\zeta) = \left(\frac{6\,z}{z^3-2}\right)^3 = \frac{6^3\,\zeta}{(\zeta-2)^3}. \tag{17}$$

The three critical points at the cube roots of -1 now coalesce into a single critical point at -1, with $g(-1) = g(8) = 8$. There is still a critical point at infinity with $g(\infty) = g(0) = 0$. But now infinity is also a critical value. In fact there if a double critical point at $\zeta = 2$, with $g(2) = \infty$. The corresponding diagram for the critical and postcritical points $2 \mapsto \infty \mapsto 0$ and $-1 \mapsto 8$ takes the form

$$\begin{array}{ccccccccc} ** & \mapsto & * & \mapsto & \odot & \qquad\qquad & * \mapsto & \odot \\ & & 3 & & 6 & & & 2 \end{array}$$

where the symbol $**$ stands for a critical point of multiplicity two. The multipliers at the two postcritical fixed points are $a^6 = -27$ and $a^2 = -3$ respectively.

Similarly we can study the Lattès map L/G_3. In this case the three points of $\mathcal{T}$ which are fixed by G_3 all map to zero. Thus the three cone points of the orbifold $\mathcal{T}/G_3$ all map to one of the three. The corresponding diagram has the following form.

$$\begin{array}{ccccccccc} ** & \mapsto & \bullet & \mapsto & \odot & \leftarrow\!\shortmid & \bullet & \leftarrow\!\shortmid & ** \\ & & 3 & & 3 & & 3 & & \end{array}$$

If we put the critical points at zero and infinity, and the postcritical fixed point at $+1$ (compare [M5]), then this map takes the form

$$f(z) = \frac{z^3 + \omega_3}{\omega_3 z^3 + 1},$$

with critical orbits $0 \mapsto \omega_3 \mapsto 1 \circlearrowleft$, and $\infty \mapsto \overline{\omega}_3 \mapsto 1 \circlearrowleft$. In contrast to L/G_2 and L/G_6, this cannot be represented as a map with real coefficients. In fact the invariant $a^3 = -i\sqrt{27}$ is not a real number, so this f is not holomorphically conjugate to its complex conjugate or mirror image map. (For a similar example with $a^3 = -8$ which occurs in the study of rational maps of the projective plane, see [BDM, §4 or §6].)

Note that f commutes with the involution $z \mapsto 1/z$. If we identify z with $1/z$ by introducing a new variable $w = z + 1/z$, then we obtain a different model for L/G_6, which is necessarily conformally conjugate to (17).

8.3. Flexible Lattès maps. Recall from §5 that there is just one connected family of flexible Lattès maps of degree a^2 for each even integer a, but that there are two distinct families of degree a^2 when a is odd. For $a^2 = 4$, the Schröder family (15), constructed by expressing $\mathrm{sn}^2(2t)$ as a rational function of $\mathrm{sn}^2(t)$, exhausts all of the possibilities. This family depends on a parameter $k^2 \in \mathbb{C}\setminus\{0, 1\}$ and has postcritical set $\{0,\ 1,\ \infty,\ 1/k^2\}$, with all postcritical points mapping to the fixed point zero. Using the corresponding formula for $\mathrm{sn}^2(3t)$ and following Schröder's method, we obtain the family

$$f(z) = \frac{z\,(k^4z^4 - 6k^2z^2 + 4(k^2+1)z - 3)^2}{(3k^4z^4 - 4k^2(k^2+1)z^3 + 6k^2z^2 - 1)^2} \tag{18}$$

of degree nine Lattès maps, with the same postcritical set $\{0,\ 1,\ \infty,\ 1/k^2\}$, but with all postcritical points fixed by f. Note that f commutes with the involution $z \mapsto 1/k^2z$ which permutes the postcritical points. Composing f with this involution, we obtain a different family

$$z \mapsto \frac{1}{k^2 f(z)} = f\left(\frac{1}{k^2 z}\right) = \frac{(3k^4z^4 - 4k^2(k^2+1)z^3 + 6k^2z^2 - 1)^2}{k^2z(k^4z^4 - 6k^2z^2 + 4(k^2+1)z - 3)^2} \tag{19}$$

with the same postcritical set, but with all postcritical orbits of period two. Higher degree examples could be worked out by the same method. Presumably they look much like the degree four case for even degrees, and much like the degree nine case for odd degrees.

Acknowledgements. I want to thank Walter Bergweiler, Serge Cantat, Alexandre Eremenko, and Curt McMullen for their help, both with the mathematics and with the history, and I want to thank the National Science Foundation (DMS 0103646) and the Clay Mathematics Institute for their support of Dynamical System activities in Stony Brook.

References

[A] D. S. ALEXANDER, "A History of Complex Dynamics", Vieweg 1994.

[Ba] C. BABBAGE. *An essay on the calculus of functions*, Phil. Trans. Royal Soc. London **105** (1815) 389–423. (Available through www.jstor.org .)

[BD] F. BERTELOOT and C. DUPONT, *Une caractérisation des endomorphismes de Lattès par leur mesure de Green*, Comm. Math. Helv. **80** (2005), 433–454.

[BL1] F. BERTELOOT and J.-J. LOEB, *Spherical hypersurfaces and Lattès rational maps*, J. Math. Pures Appl. **77** (1998) 655–666.

[BL2] F. BERTELOOT and J.-J. LOEB, *Une caractérisation géométrique des exemples de Lattès de* $\mathbf{P}^k$, Bull. Soc. Math. France **129** (2001), 175–188.

[BM] F. BERTELOOT and V. MAYER, "Rudiments de dynamique holomorphe", Soc. Math. France 2001.

[BDM] A. BONIFANT, M. DABIJA and J. MILNOR, *Elliptic curves as attractors in* $\mathbb{P}^2$*, Part 1, in preparation.*

[Bö] L. E. BÖTTCHER, *The principal laws of convergence of iterates and their application to analysis* (Russian), Izv. Kazan. Fiz.-Mat. Obshch. **14** (1904) 155–234.

[De] L. DEMARCO, *Iteration at the boundary of the space of rational maps*, Preprint, U. Chicago 2004.

[Di1] T.-C. DINH, *Sur les applications de Lattès de* $\mathbb{P}^k$, J. Math. Pures Appl. **80** (2001) 577–592.

[Di2] ——, *Sur les endomorphismes polynomiaux permutables de* $\mathbb{C}^2$, Ann. Inst. Fourier **51** (2001) 431–439.

[DS] T.-C. DINH and N. SIBONY, *Sur les endomorphismes holomorphes permutables de* $\mathbb{P}^k$, Math. Annalen **324** (2002) 33–70.

[DH] A. DOUADY and J. HUBBARD, *A proof of Thurston's topological characterization of rational functions*, Acta. Math. **171** (1993) 263–297.

[DMc] P. DOYLE and C. MCMULLEN, *Solving the quintic by iteration*, Acta Math. **163** (1989) 151–180.

[Du] C. DUPONT, Exemples de Lattès et domaines faiblement sphériques de $\mathbb{C}^n$, Manuscripta Math. **111** (2003) 357–378.

[E] A. EREMENKO, *On some functional equations connected with iteration of rational functions*, Algebra i Analiz **1** (1989) 102–116 (Russian); Leningrad Math. J. (now St. Peterburg Math. J.) **1** (1990) 905–919 (English).

[FLM] A. FREIRE, A. LOPES, and R. MAÑÉ, *An invariant measure for rational maps*, Bol. Soc. Brasil. Mat. **14** (1983), 45–62.

[H] M. HERMAN, *Examples de fractions rationnelles ayant une orbite dense sur la sphère de Riemann*, Bull. Sci. Math. France **112** (1984) 93–142.

[K] G. KŒNIGS, *Recherches sur les integrals de certains equations fonctionelles*, Ann. Sci. Éc. Norm. Sup. **1** (1884) suppl. 1–41.

[La] S. LATTÈS, *Sur l'iteration des substitutions rationnelles et les fonctions de Poincaré*, C. R. Acad. Sci. Paris **166** (1918) 26–28. (See also ibid. pp. 486–89.)

[LP] G. LEVIN and F. PRZYTYCKI, *When do two rational functions have the same Julia set?*, Proc. A. M. S. **125** (1997) 2179–2190.

[Ly] M. LYUBICH, *Entropy properties of rational endomorphisms of the Riemann sphere*, Ergodic Th. and Dy. Sys. **3** (1983) 351–385.

[Ma] R. MAÑÉ, *On the uniqueness of the maximizing measure for rational maps.* Bol. Soc. Brasil. Mat. **14** (1983) 27–43.

[MSS] R. MAÑÉ, P. SAD, and D. SULLIVAN, *On the dynamics of rational maps*, Ann. Sci. École Norm. Sup. **16** (1983) 193–217.

[Mc1] C. MCMULLEN, *Families of rational maps and iterative root-finding algorithms*, Ann. of Math. **125** (1987) 467–493.

[Mc2] ——, *Frontiers in complex dynamics*, Bull. A. M. S. **31** (1994) 155–172.

[Mc3] ——, "Complex Dynamics and Renormalization", Annals Math. Studies **135**, Princeton U. Press 1994.

[M1] J. MILNOR, *On the 3-dimensional Brieskorn manifolds* $M(p,q,r)$, in "Knots, Groups and 3-manifolds, Annals Math. Studies **84**, Princeton Univ. Press 1975, pp. 175–224; or in Milnor "Collected Papers **2**", Publish or Perish 1995.

[M2] ——, *Hilbert's Problem 18: on crystallographic groups, fundamental domains, and on sphere packing* , Proceedings Symposia in Pure Math. **28**, part 2, Amer. Math. Soc. 1976, pp. 491–506; or in Milnor "Collected Papers **1**", Publish or Perish 1994.

[M3] ——, *Geometry and dynamics of quadratic rational maps*, Experimental Math. **2** (1993) 37–83.

[M4] ——, "Dynamics in One Complex Variable", Vieweg, 1999, 2000; third edition: Princeton University Press 2006, to appear.

[M5] ——, *Rational maps with two critical points*, Experimental Math. **9** (2000) 481–522.

[M6] ——, *Pasting together Julia sets*, a worked out example of mating, Experimental Math. **13** (2004) 55–92.

[P] H. POINCARÉ, *Sur une classe nouvelle de transcendantes uniformes*, J. Math. Pures Appl. **6** (1890) 313–365 (Œvres IV, 537–582).

[Re] M. REES, *Positive measure sets of ergodic rational maps.*, Ann. Sci. École Norm. Sup. **19** (1986) 383–407.

[R1] J. F. RITT, *Periodic functions with a multiplication theorem*, Trans. Amer. Math. Soc. **23** (1922) 16–25.

[R2] ——, *Permutable rational functions* , Trans. Amer. Math. Soc. **25** (1923) 399–448.

[R3] ——, *Meromorphic functions with addition or multiplication theorems*, Trans. Amer. Math. Soc. 29 (1927), 341–360.

[S] E. SCHRÖDER, *Ueber iterirte Functionen*, Math. Ann. **3** (1871), 296–322. Available through http://gdz.sub.uni-goettingen.de/en (search by title).

[V] A. P. VESELOV, *Integrable maps*, Uspekhi Mat. Nauk **46:5** (1991) 3–45; Russ. Math. Surveys **46:5** (1991) 1–51.

[WW] E. T. WHITTAKER and G. N. WATSON, "A Course of Modern Analysis", Cambridge U. Press 1966 (4th edition).

[Z] A. ZDUNIK, *Parabolic orbifolds and the dimension of the maximal measure for rational maps*, Inv. Math. **99** (1990) 627–649.

Revised July 2005

Institute for Mathematical Sciences
Stony Brook University
Stony Brook,
NY 11794-3660 USA
e-mail: jack@math.sunysb.edu
www.math.sunysb.edu/∼jack

Branner-Hubbard motions and attracting dynamics

C. L. Petersen
Tan Lei

Abstract. Branner-Hubbard motion is a systematic way of deforming an attracting holomorphic dynamical system f into a family $(f_s)_{s \in \mathbb{L}}$, via a holomorphic motion which is also a group action. We establish the analytic dependence of f_s on s (a result first stated by Lyubich) and the injectivity of f_s on f. We prove that the stabilizer of f (in terms of s) is either the full group $\mathbb{L}$ (rigidity), or a discrete subgroup (injectivity). The first case means that f_s is Möbius conjugate to f for all $s \in \mathbb{L}$, and it happens for instance at the center of a hyperbolic component. In the second case the map $s \mapsto f_s$ is locally injective. We show that BH-motion induces a periodic holomorphic motion on the parameter space of cubic polynomials, and that the corresponding quotient motion has a natural extension to its isolated singularity. We give another application in the setting of Lavaurs enriched dynamical systems within a parabolic basin.

1. Introduction. This paper describes systematic perturbations of holomorphic dynamical systems via structured holomorphic motions that are also group actions. The technique, commonly known as the Branner Hubbard motion or in short BH-motion, was introduced by Branner and Hubbard in [BH1] to study parameter spaces of monic polynomials. It was later used by, for example, Branner's Ph.D. student Willumsen [Wi] (see also [Ta] in this volume).

In order to better exhibit the general properties of BH-motions we introduce the notion of attracting dynamics (see Definition 2.3), on which the BH-motion naturally acts. Given an attracting dynamics f, a BH-motion provides a parametrized family of attracting dynamics $(f_s)_{s \in \mathbb{L}}$, with the parameter space $\mathbb{L}$ equal to the right half complex plane, equipped with its usual complex structure and with a specific real Lie group structure. The map $s \mapsto f_s$ is naturally a group action.

We give a thorough description of the construction of the BH-motion together with basic properties. We prove then the holomorphic dependence of f_s on s (Theorem 2.5.(2)), in a holomorphic motion context more general than BH-motions (Theorem 2.7). This result was first stated without proof by Michael Lyubich. We proceed to prove the injectivity of $f \mapsto f_s$ (Theroem 2.5.(6)). These two results will be our main tool, while performing BH-motions on a full slice of cubic polynomial attracting dynamics, to promote a holomorphic motion of the dynamical planes to a holomorphic motion of the parameter space of such polynomials.

We then study the mapping properties of $s \mapsto f_s$. We show that the stabilizer (see Definition 3.1) exhibits the following dichotomy: it is

• either the full group $\mathbb{L}$, in which case f behaves like the center of a hyperbolic component, in other words all critical points attracted by the attracting cycle actually lay on the cycle or its preimages;
• or a discrete subgroup of $\mathbb{L}$ contained in the vertical line $1 + i\mathbb{R}$, in which case f is necessarily a non-center, in other words at least one attracted critical point has an infinite orbit (Theorem 3.3).

There are many possible applications of BH-motions. We have chosen here two of them which we find illustrative for the diversity of applications.

The first one concerns a family (P_a) of cubic polynomials, such that 0 is an attracting fixed point of multiplier independent of a, and attracts exactly one simple critical point. We will perform a BH-motion on each P_a, thus obtain a double indexed family $P_{a,s}$ of cubic polynomials. As mentioned above, we prove that these dynamically defined BH-motions promote to a holomorphic motion of the a-slice within the space of cubic polynomials, which turns out to be $2\pi i$-periodic on s (Theorem 4.1). This induces naturally a quotient motion over $\mathbb{D}^* \approx \mathbb{L}/2\pi i\mathbb{Z}$, which is the most natural way to change the multiplier of P_a at 0, but keeping the remaining part of the dynamics fixed. We then make one more effort to extend this motion over $\mathbb{D}$, and thus succeed in deforming systematically the attracting fixed point into a superattracting fixed point (Theorem 4.2).

The second one concerns the BH-motion of the basin of ∞ of the quadratic cauliflower $z \mapsto z^2 + \frac{1}{4}$, enriched by a Lavaurs map g. We give a detailed study of the effect of the BH-motion on the enrichment of the dynamics (Theorem 4.6).

For other illustrations beyond the paper of Willumsen and the original paper of Branner and Hubbard, the reader may want to consult the beautiful master thesis of Uhre [U], the paper [D] in this volume and the paper [P-T] which explores further the notion of attracting dynamics.

2. Definition and basic properties of BH-motions. In this paper we shall use the notion of holomorphic motions in a slightly more general sense than the usual definition:

Definition 2.1 (Holomorphic motion)**.** Let (χ, Λ, p) be a triple with χ, Λ two complex analytic manifolds and $p : \chi \to \Lambda$ an analytic surjective mapping. Denote by χ_λ the fiber $p^{-1}(\lambda)$. Choose $\lambda_0 \in \Lambda$ a **base point** and $E \subset \chi_{\lambda_0}$. A **holomorphic motion** of E over Λ into χ is a mapping $H : \Lambda \times E \to X$, $(\lambda, z) \mapsto H(\lambda, z)$ satisfying:

1. For any fixed λ, $z \mapsto H(\lambda, z)$ is injective on E and maps E into χ_λ.
2. For any fixed $z \in E$, $\lambda \mapsto H(\cdot, z)$ is analytic.
3. $H(\lambda_0, \cdot)$ is the identity map on E.

In practice, we often have $\chi = \Lambda \times X$, in which case we suppress the first coordinate of H and write $H : \Lambda \times E \longrightarrow X$.

2.1. The model BH-motion. The notation below is taken from [Wi]. Further calculations can be found there.

Define $\mathbb{L} = \{u + iv,\, u > 0\}$ and for any $s = s_x + is_y \in \mathbb{L}$, define an $\mathbb{R}$-linear diffeomorphism $\widetilde{l_s} : \mathbb{C} \to \mathbb{C}$ by:

$$\widetilde{l_s}(z) = (s-1)z_x + z = sz_x + iz_y = \frac{s+1}{2} z + \frac{s-1}{2} \bar{z}$$
$$= \begin{pmatrix} s_x & 0 \\ s_y & 1 \end{pmatrix} \begin{pmatrix} z_x \\ z_y \end{pmatrix} = \begin{pmatrix} s_x z_x \\ s_y z_x + z_y \end{pmatrix};$$

where $z = z_x + iz_y$. Moreover define a homeomorphism $l_s : \mathbb{C} \to \mathbb{C}$ by

$$l_s(z) = l_s(re^{2\pi i\theta}) = r^s e^{2\pi i\theta} = z \cdot r^{s-1} = z \cdot e^{(s-1)\log r},$$

so that $\exp \circ \widetilde{l_s} = l_s \circ \exp$. Then the almost complex structure $\widetilde{\sigma_s} = \widetilde{l_s}^*(\sigma_0)$ obtained by pulling back the standard almost complex structure σ_0, is given by the 'constant' Beltrami form $t_s \frac{d\bar{z}}{dz}$ where the constant $t_s = \frac{s-1}{s+1}$ depends only on s, but not the position z.

Let $\star : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ denote the group structure for which the map $s \mapsto \widetilde{l_s}$, is a group isomorphism onto its image, i.e., $s' \star s \mapsto \widetilde{l_{s'}} \circ \widetilde{l_s}$ and in algebra

$$s' \star s = \frac{s'(s + \bar{s}) + (s - \bar{s})}{2} = s'_x s_x + i(s'_y s_x + s_y), \text{ and}$$
$$s^{-1} = \frac{1}{s_x} - i\frac{s_y}{s_x} = \frac{2 - s + \bar{s}}{s + \bar{s}}.$$

Note that $s = (s_x + i0) \star (1 + is_y)$. The group $(\mathbb{L}, \star)$ is therefore a real Lie group and is generated by the two Abelian but non mutually commuting subgroups $(\mathbb{W}, \star)$ and $(\mathbb{S}, \star)$ called wring and stretch respectively. Where $\mathbb{W} = 1 + i\,\mathbb{R}$ with $\star$ is given by addition of imaginary parts and where $\mathbb{W}$ acts on the group $(\mathbb{L}, \star)$ from the right by addition of the imaginary part. And $\mathbb{S} = \mathbb{R}_+$ with $\star$ given by multiplication and where $\mathbb{S}$ acts on the group $(\mathbb{L}, \star)$ from the right by multiplication. The (collection of) maps $\widetilde{l_s}$ and l_s have many useful properties. We state them below as:

Define $\mathbb{H}_\pm = \{z \in \mathbb{C} | \pm \Re(z) > 0\}$, the right (left) half plane.

Lemma 2.2 (basic properties of $\widetilde{l_s}$ and l_s)**.**

1. *The* map $\widetilde{l_s}$ *is the unique linear map mapping the ordered triple* $(i, 0, 1)$ *to the ordered triple* $(i, 0, s)$*. The maps*

$$(s, z) \mapsto \widetilde{l_s}(z), \quad \mathbb{L} \times \mathbb{C} \to \mathbb{C}, \quad \mathbb{L} \times \mathbb{H}_\pm \to \mathbb{H}_\pm$$

$$(s, z) \mapsto l_s(z), \quad \mathbb{L} \times \mathbb{C} \to \mathbb{C}$$

satisfy simultaneously the following properties:

- they are **holomorphic motions** *over* $\mathbb{L}$*, with base point* $s_0 = 1$
- they are ***group actions*** of $(\mathbb{L}, \star)$, *acting on the left.*

- they are ***dynamical conjugacies***, *more precisely* $\widetilde{l}_s$ *conjugates* $z \mapsto kz$ $(k \in \mathbb{R})$ *to itself and conjugates* $z \mapsto z + L$ *to* $z \mapsto z + L_s$, *where* $L_s := \widetilde{l}_s(L)$; *and* l_s *conjugates* $z \mapsto z^k$ $(k \in \mathbb{N})$ *to itself and conjugates* $z \mapsto \lambda z$ *to* $z \mapsto \lambda_s z$, *where* $\lambda_s := l_s(\lambda) = \lambda \cdot |\lambda|^{s-1}$.

2. $\widetilde{l}_s|_{i\mathbb{R}} = Id$, *it maps* $\mathbb{R}$ *to the oblique line passing through* 0 *and* s, *and any other horizontal line* $iy + \mathbb{R}$ *to the line parallel to* s, *passing through* iy. *It maps any vertical line to a vertical line.*
3. *For any* $z \in \mathbb{H}_\pm$: $\mathrm{d}_{\mathbb{H}_\pm}(z, \widetilde{l}_s(z)) = \mathrm{d}_{\mathbb{H}_+}(1, s) = C_s$, *where* $\mathrm{d}_{\mathbb{H}_\pm}$ *denotes the hyperbolic distance.*
4. *For* $s_1 = 1 + 2\pi i$, $\widetilde{l}_{s_1}$ maps $m + i \cdot \mathbb{R}$ $(m \in \mathbb{Z})$ *onto itself, mapping* m *to* $m + 2\pi i m$.
5. $l_s|_{\{|z|=1\}} = id$, l_s *maps the circle* $|z| = r$ *to the circle* $\{|z| = r^u\}$ (*where* $u = \Re(s)$), *and the ray* $e^{2\pi i\theta} \cdot \mathbb{R}_+$ *to an oblique* (*logarithmic*) *ray.*
6. *The Beltrami coefficient of* $\widetilde{l}_s(z)$ *is a constant depending only on* s, *i.e., is translationally invariant and that of* l_s *is invariant under linear maps* $z \mapsto az$, $a \neq 0$:
$$\frac{\overline{\partial}\widetilde{l}_s}{\partial \widetilde{l}_s}(z) \equiv \frac{s-1}{s+1} =: t_s \in \mathbb{D}, \qquad \frac{\overline{\partial} l_s}{\partial l_s}(z) \equiv \frac{z}{\bar{z}} \frac{s-1}{s+1} =: t_s \in \mathbb{D}.$$
Moreover the dilatations $K(l_s) = K(\widetilde{l}_s) = \frac{1+|t|}{1-|t|}$ *are also constants.*
7. *When* s *varies from* 1 *to* $1 + 2\pi i$, *the circle* $\{|z| = e^m\}$ *makes* m*-turns, relative to the unit circle, for all* $m \in \mathbb{Z}$, *under the action of* l_s.

Proof. We will only prove 3 for $\mathbb{H}_+$, the rest being straight forward. Fix $z_0 \in \mathbb{H}_+$. The map $w \mapsto w - i \cdot \Im(z_0)$ is a hyperbolic isometry of $\mathbb{H}_+$, mapping z_0 to $\Re(z_0)$ and $\widetilde{l}_s(z_0)$ to $\widetilde{l}_s(\Re(z_0))$ by the conjugacy property). Now $w \mapsto w/\Re(z_0)$ is again an isometry, mapping $\Re(z_0)$ to 1 and $\widetilde{l}_s(\Re(z_0))$ to $\widetilde{l}_s(1) = s$ (by the conjugacy property). □

2.2. BH-motion of an attracting dynamics (f, W, α).

Definition 2.3. We say that (f, W, α) or in short (f, α) or f, is an **attracting dynamics**, if: i) $W \subset \bar{\mathbb{C}}$ is open, ii) $f : W \longrightarrow \bar{\mathbb{C}}$ is holomorphic and iii) $\alpha \in W$ is an attracting or superattracting periodic point for f.

Any attracting dynamics (f, W, α) comes with a **long string** of informations

$$(f, W, \alpha, k(f), \lambda(f), \widetilde{B}(\alpha), B(\alpha), \phi, U)$$

defined as follows: $k = k(f) \in \mathbb{N}$ is the exact period of α, and $\lambda(f) \in \mathbb{D}$ denotes the multiplier $(f^k)'(\alpha)$. The set

$$\widetilde{B}(\alpha) := \{z \in W | \ \forall n \ \ f^n(z) \in W \ \& \ f^{nk+l}(z) \underset{n \to +\infty}{\longrightarrow} \alpha \text{ for some } l \in \mathbb{N}\}$$

denotes the entire attracted basin of the orbit of α, and $B(\alpha)$ denotes the immediate basin of α, i.e., the connected component of $\widetilde{B}(\alpha)$ containing α. The map $\phi : U \longrightarrow \mathbb{C}$

is a choice of a linearizing (*possibly a Böttcher*) *coordinate defined and univalent on some neighborhood U of* α.

Note that different choices of ϕ on a given U differ by a multiplicative constant. In what follows we shall for any subset $W \subseteq \bar{\mathbb{C}}$ denote by W^c the complement $\bar{\mathbb{C}} \setminus W$.

Definition 2.4. Define a **BH-motion** of (f, W, α) to be a map:

$$s \mapsto (\sigma_s, h_s, \ (f_s, W_s, \alpha_s), \ \phi_s, U_s), s \in \mathbb{L} \qquad \text{or in short} \quad s \mapsto h_s, s \in \mathbb{L}$$

as follows (see the diagram below):

- σ_s is the measurable and bounded Beltrami form defined by

$$\sigma_s = \sigma_{s,\,f} = \begin{cases} (l_s \circ \phi)^*(\sigma_0) & \text{on } U \\ (f^n)^* \sigma_s & \text{on } f^{-n}(U), n \in \mathbb{N} \\ \sigma_0 & \text{on } \widetilde{B}(\alpha)^c. \end{cases}$$

That is σ_s is given by the above formulas on U and $\widetilde{B}(\alpha)^c$ and extended to $\widetilde{B}(\alpha)$ by iterated pull-backs of f. Note that for every $z_0 \in U$ the assignment $s \mapsto \sigma_s(z_0)$ is a complex analytic function on $\mathbb{L}$. In fact if we write $\sigma_s(z) = \mu_s(z) \frac{d\bar{z}}{dz}$ in some local coordinate z on $W' \subseteq W$, then for every fixed $z_0 \in U$ the map $s \mapsto \mu_s(z_0) : \mathbb{L} \longrightarrow \mathbb{D}$ is a Möbius transformation. On $\mathbb{C}$ which has a natural preferred chart the identity, we shall abuse the notation and simply write μ for the Beltrami form $\mu \frac{d\bar{z}}{dz}$.

- $h_s = h_{s,\,f} : \bar{\mathbb{C}} \longrightarrow \bar{\mathbb{C}}$ is a family of **integrating maps** for σ_s normalized so as to depend complex analytically on s, as supplied by the measurable Riemann mapping theorem with parameters.
- $(f_s, W_{f_s}, \alpha_{f_s}) = (f_s, W_s, \alpha_s) = (h_s \circ f \circ h_s^{-1}, \ h_s(W), h_s(\alpha))$.
- $\phi_s = \phi_{f_s} = l_s \circ \phi \circ h_s^{-1}$ and $U_s = U_{f_s} = h_s(U)$.

Note that another s-analytic normalization $\widehat{h_s}$ of h_s differs by an s-analytic family of Möbius transformations. In other words $\widehat{h_s} = M_s \circ h_s$ with M_s Möbius and analytic in s.

Example 1. $(f, W, \alpha) = (e^{-1}z + z^2, \bar{\mathbb{C}}, 0)$ is an attracting dynamics with $k(f) = 1$ and $\lambda(f) = e^{-1}$. In a BH-motion of it, we may normalize the integrating maps h_s so that they fix 0 and ∞ and they are tangent to the identity at ∞. Then one checks easily that $f_s(z) = e^{-s}z + z^2$.

Example 2. Let P be a monic centered polynomial. We do a BH-motion for $(f, W, \alpha) = (P, \bar{\mathbb{C}}, \infty)$. We normalize h_s so that each f_s is again monic centered. In case that P has a connected Julia set, a theorem of Branner-Hubbard ([BH], *Propostion* 8.3) *shows that* $f_s \equiv P$. *We will reprove this result below in a more general setting.*

The basic properties of a BH-motion are:

Theorem 2.5 (BH-motion of dynamics)**.** *Let* (f, W, α) *be an attracting dynamics with* $k(f) = k$ *and* $\lambda(f) = \lambda$ *and let* $s \mapsto (\sigma_s, h_s, \ (f_s, W_s, \alpha_s), \phi_s, U_s)$ *be a BH-motion of* (f, W, α). *Then:*

1. *For any* $z \in \bar{\mathbb{C}}$, *the assignment* $s \mapsto \sigma_s(z)$ *is independent of the choices of* (ϕ, U) *in the long string information.*
2. *The two maps of two complex variables* $(s, z) \mapsto f_s(z)$ *and* $(s, z) \mapsto \phi_s(z)$ *are complex analytic in* $\{(s, z), s \in \mathbb{L}, z \in W_s\}$ *and* $\{(s, z), s \in \mathbb{L}, z \in U_s\}$ *respectively.*
3. *For any* $s \in \mathbb{L}$ *the triple* (f_s, W_s, α_s) *is again an attracting dynamics, whose long string of information takes the form*:

$$(f_s, W_s, \alpha_s, k, \lambda|\lambda|^{s-1}, h_s(\widetilde{B}(\alpha)), h_s(B(\alpha)), \phi_s, U_s).$$

If $\lambda \in D^\star$ *then* $s \mapsto \lambda_s$ *is holomorphic and depends on* λ *only.*
If $\lambda = 0$ *then* $\lambda_s \equiv 0$ *and* ϕ_s *is a Böttcher coordinate for* (f_s, α_s), *defined and univalent on* U_s. *Moreover write* $f^k(z) - \alpha = a(z-\alpha)^d + \text{higher order terms}$, *with* $a \neq 0$ *and* $d > 1$ *the local degree at* α, *then* $f_s^k(z) - \alpha_s = a(s)(z-\alpha_s)^d + \text{higher order terms}$, *with* $a(s)$ *non vanishing and holomorphic in s.*
4. *If* ϕ *extends holomorphically to some domain* $U' \subset \widetilde{B}(\alpha)$ *then* $l_s \circ \phi \circ h_s^{-1}$ *is a holomorphic extension of* ϕ_s *to* $U'_s = h_s(U') \subset \widetilde{B}(\alpha_s)$.
5. *The maps* $(s, z) \mapsto h_s(z)$
   - *form a* ***holomorphic motion*** *of* $\bar{\mathbb{C}}$ *over* $\mathbb{L}$ *with base point* $s = 1$;
   - *are* ***dynamical conjugacies*** *as indicated in the following diagram:*

$$\begin{array}{ccccc} (f) & & (\lambda z, z^m) & & (z+L, mz) \\ W \supset \widetilde{B}(\alpha) \supset U & \xrightarrow{\phi} & \phi(U) \subset \mathbb{C} & \xleftarrow{\exp} & \mathbb{C},\ (i, 0, 1) \\ \big\downarrow h_s & & \big\downarrow l_s & & \big\downarrow \widetilde{l}_s \text{ linear} \\ W_s \supset \widetilde{B}(\alpha_s) \supset U_s & \xrightarrow[\phi_s := l_s \phi h_s^{-1}]{} & \phi_s(U_s) \subset \mathbb{C} & \xleftarrow[\exp]{} & \mathbb{C},\ (i, 0, s) \\ (f_s) & & (\lambda_s z, z^m) & & (z+L_s, mz) \end{array} \quad (1)$$

   - *are* ***group actions***. *More precisely,* $(f_s)_{s'} = f_{s' \star s}$, $h_{s' \star s,\, f} = h_{s',\, f_s} \circ h_{s,\, f}$ *and* $\phi_{s' \star s,\, f} = (\phi_{s,\, f})_{s',\, f_s}$, (*subject to suitable normalizations*). *And for any fixed* $s \in \mathbb{L}$, *the map*

$$s' \mapsto (\sigma_{s' \star s}, h_{s' \star s}, (f_{s' \star s}, W_{s' \star s}, \alpha_{s' \star s}), \phi_{s' \star s}, U_{s' \star s})$$

   is a BH-motion of f_s.

6. (**injectivity**) *If* $(f_{s_0}, W_{f_{s_0}}, \alpha_{f_{s_0}})$ *and* $(g_{s_0}, W_{g_{s_0}}, \alpha_{g_{s_0}})$ *are Möbius conjugate by M* (*see Definition* 2.6 *below*) *for some* $s_0 \in \mathbb{L}$ *and some pair of attracting dynamics* (f, W_f, α_f) *and* (g, W_g, α_g). *Then* $(f_s, W_{f_s}, \alpha_{f_s})$ *and* $(g_s, W_{g_s}, \alpha_{g_s})$ *are Möbius conjugate for all s by a holomorphically varying family* M_s *of Möbius transformations with* $M_{s_0} = M$.

The proof of this theorem is postponed to the next subsection.

Definition 2.6. Two attracting dynamics (f, W_f, α_f) and (g, W_g, α_g) are Möbius **conjugate**, if there is a Möbius transformation M, with $M(\alpha_f) = \alpha_g$, $M(W_f) = W_g$ and $M \circ f = g \circ M$.

Remark. The map $s \mapsto f_s$ may not extend continuously to the boundary $i\mathbb{R}$ of $\mathbb{L}$. See [BH1], [Wi], [KN] or [Ta] for details.

2.3. Proof of Theorem 2.5. The non trivial part of Theorem 2.5 is the analytic dependence on s of $f_s(z)$ and $\phi_s(z)$. It is a consequence of a theorem first stated by Lyubich, which we restate and prove below. It requires however a little setup.

Let $U, V \subset \bar{\mathbb{C}}$ be open subsets and $f : U \longrightarrow V$ be a holomorphic map. Let Λ be a complex analytic manifold and suppose $\sigma : \Lambda \times V \longrightarrow \mathrm{Bel}\,(V)$ is an analytically varying family of bounded measurable Beltrami forms supported on V. Let $\Psi : \Lambda \times \bar{\mathbb{C}} \longrightarrow \bar{\mathbb{C}}$ be a corresponding analytically varying family of integrating quasi-conformal homeomorphisms as supplied by the measurable Riemann mapping theorem with parameters. That is for each fixed $z \in \bar{\mathbb{C}}$: $\lambda \mapsto \Psi_\lambda(z)$ is holomorphic and for each λ the map $\Psi_\lambda = \Psi(\lambda, \cdot)$ is a quasi-conformal homeomorphism with $\Psi_\lambda^*(\sigma_0) = \sigma_\lambda$ on V and $\Psi_\lambda^*(\sigma_0) = \sigma_0$ on V^c. Let similarly $\Phi : \Lambda \times \bar{\mathbb{C}} \longrightarrow \bar{\mathbb{C}}$ be an analytically varying family of integrating quasi-conformal homeomorphisms $\Phi_\lambda = \Phi(\lambda, \cdot)$ for the pulled-back structures $\widehat{\sigma}_\lambda = f^*(\sigma_\lambda)$, with $\Phi_\lambda^*(\sigma_0) = \widehat{\sigma}_\lambda$ on U and $\Phi_\lambda^*(\sigma_0) = \sigma_0$ on U^c. See diagram (2).

$$\begin{array}{ccc} (U, \widehat{\sigma}_\lambda) & \xrightarrow{\Phi_\lambda} & (U_\lambda, \sigma_0) \\ f\big\downarrow & & \big\downarrow f_\lambda \\ (V, \sigma_\lambda) & \xrightarrow{\Psi_\lambda} & (V_\lambda, \sigma_0) \end{array} \tag{2}$$

Define for each $\lambda \in \Lambda$: $U_\lambda = \Phi_\lambda(U)$ and $V_\lambda = \Psi_\lambda(V)$ and open subsets $\mathcal{U}, \mathcal{V} \subseteq \Lambda \times \bar{\mathbb{C}}$ by $\mathcal{U} = \{(\lambda, z) | z \in U_\lambda\}$ and $\mathcal{V} = \{(\lambda, z) | z \in V_\lambda\}$. Finally define a continuous map (homeomorphism if f is bi holomorphic) $\mathrm{F} : \mathcal{U} \longrightarrow \mathcal{V}$ by

$$F(\lambda, z) = (\lambda, f_\lambda(z)) := (\lambda, \Psi_\lambda \circ f \circ \Phi_\lambda^{-1}(z)).$$

Note that although $\Phi_\lambda^{-1}(z)$ is still quasi-conformal in z and continuous in (λ, z), it is in general no more analytic in λ. However, we have, as a miracle,

Theorem 2.7 (Lyubich). *The above map F is complex analytic or equivalently $(\lambda, z) \mapsto f_\lambda(z)$ is complex analytic.*

Proof. The map $(\lambda, z) \mapsto f_\lambda(z) : \mathcal{U} \longrightarrow \bar{\mathbb{C}}$ is continuous, because the two maps

$$(\lambda, z) \mapsto (\lambda, \Phi_\lambda(z)) : \Lambda \times U \longrightarrow \mathcal{U} \quad \text{and} \quad (\lambda, z) \mapsto (\lambda, \Psi_\lambda(z)) : \Lambda \times V \longrightarrow \mathcal{V}$$

are homeomorphisms. Moreover f_λ is holomorphic for each fixed λ_0 as f_λ pulls back the standard Beltrami form σ_0 to itself,

$$f_\lambda^*(\sigma_0) = (\Psi_\lambda \circ f \circ \Phi_\lambda^{-1})^*(\sigma_0) = (\Phi_\lambda^{-1})^* \circ f^* \circ \Psi_\lambda^*(\sigma_0) = (\Phi_\lambda^{-1})^*(f^*(\sigma_\lambda)) = \sigma_0.$$

Thus we need only check that for each fixed z the map $\lambda \mapsto f_\lambda(z)$ is holomorphic in each of the coordinate functions of a complex analytic local chart on Λ. Equivalently we need to prove that $f_\lambda(z)$ has a complex partial derivative at each point $(\lambda_0, z_0) \in \mathcal{U}$ with respect to each such coordinate function. Hence the theorem is an immediate consequence of the following one variable version. $\square$

Proposition 2.8. *In the notation above suppose* $\Lambda, U, V \subset \mathbb{C}$ *are open subsets and* $\mathcal{U}, \mathcal{V} \subseteq \Lambda \times \mathbb{C}$. *For* $(\lambda_0, z_0) \in \mathcal{U}$ *write* $w_0 = \Phi_{\lambda_0}^{-1}(z_0) \in U$ *then*

$$\frac{\partial f_\lambda}{\partial \lambda}(z_0)\Big|_{\lambda=\lambda_0} := \lim_{\lambda \to \lambda_0} \frac{f_\lambda(z_0) - f_{\lambda_0}(z_0)}{\lambda - \lambda_0} = \frac{\partial \Psi_\lambda}{\partial \lambda}(f(w_0))\Big|_{\lambda=\lambda_0} - f'_{\lambda_0}(z_0) \cdot \frac{\partial \Phi_\lambda}{\partial \lambda}(w_0)\Big|_{\lambda=\lambda_0}.$$

The same proof shows that if λ is a real parameter and $\Phi_\lambda(w_0)$ and $\Psi_\lambda(f(w_0))$ are real partially differentiable as functions of λ at λ_0, then $f_\lambda(z_0)$ is partially real differentiable at λ_0 with the same formula for the partial derivative. See the following diagram:

$$\begin{array}{ccc} w_0 & \xrightarrow{\Phi_{\lambda_0}} & z_0 \\ f\big\downarrow & & \big\downarrow f_{\lambda_0} \\ f(w_0) & \xrightarrow{\Psi_{\lambda_0}} & f_{\lambda_0}(z_0) \end{array}$$

Proof. At first $(\lambda, z) \mapsto f_\lambda(z)$ is continuous on (λ, z) and analytic on z. By the Cauchy integral formula, $f'_\lambda(z)$ and $f''_\lambda(z)$ depend continuously on (λ, z). In particular, for

$$u(\lambda, z) := f_\lambda(z) - f_\lambda(z_0) - f'_\lambda(z_0)(z - z_0),$$

there is some $\kappa > 0$ such that $|u(\lambda, z)| \le \kappa |z - z_0|^2$ for (λ, z) in a neighborhood of (λ_0, z_0).

As the maps $\lambda \mapsto \Phi_\lambda(z_0)$ and $\lambda \mapsto \Psi_\lambda(f(w_0))$ are $\mathbb{C}$-differentiable at λ_0, we can write $\frac{\partial \Phi_\lambda}{\partial \lambda}(w_0)|_{\lambda_0} = A$, $\frac{\partial \Psi_\lambda}{\partial \lambda}(f(w_0))|_{\lambda_0} = B$ and $\Phi_\lambda(w_0) - z_0 = A(\lambda - \lambda_0) + o(|\lambda - \lambda_0|)$. From the relation $f_\lambda \circ \Phi_\lambda = \Psi_\lambda \circ f$ we obtain

$$\begin{aligned} B \underset{\lambda \to \lambda_0}{\longleftarrow} \frac{\Psi_\lambda(f(w_0)) - \Psi_{\lambda_0}(f(w_0))}{\lambda - \lambda_0} &= \frac{f_\lambda(\Phi_\lambda(w_0)) - f_{\lambda_0}(\Phi_{\lambda_0}(w_0))}{\lambda - \lambda_0} \\ &= \frac{f_\lambda(\Phi_\lambda(w_0)) - f_{\lambda_0}(z_0)}{\lambda - \lambda_0} \\ &= \frac{f_\lambda(\Phi_\lambda(w_0)) - f_\lambda(z_0)}{\lambda - \lambda_0} + \frac{f_\lambda(z_0) - f_{\lambda_0}(z_0)}{\lambda - \lambda_0} \\ &= \frac{f'_\lambda(z_0)(\Phi_\lambda(w_0) - z_0) + u(\lambda, \Phi_\lambda(w_0))}{\lambda - \lambda_0} \\ &\quad + \frac{f_\lambda(z_0) - f_{\lambda_0}(z_0)}{\lambda - \lambda_0} \\ &= \frac{f'_\lambda(z_0)A(\lambda - \lambda_0) + o(|\lambda - \lambda_0|)}{\lambda - \lambda_0} \\ &\quad + \frac{f_\lambda(z_0) - f_{\lambda_0}(z_0)}{\lambda - \lambda_0}. \end{aligned}$$

It follows that

$$\lim_{\lambda\to 0}\frac{f_\lambda(z_0)-f_{\lambda_0}(z_0)}{\lambda-\lambda_0}=B-f'_{\lambda_0}(z_0)\cdot A.$$

□

Proof of Theorem 2.5.

1. Let (ϕ_i, U_i), $i=1,2$ be two choices of the linearizer (or Böttcher coordinate). Then on a neighborhood $U\subset U_1\cap U_2$ of α, there is a constant $a\neq 0$ such that $\phi_1=a\cdot\phi_2$. But $(a\cdot)^*(l_s^*\sigma_0)=l_s^*\sigma_0$, by 6. of Lemma 2.2. Thus on U, $(l_s\circ\phi_1)^*\sigma_0=(l_s\circ\phi_2)^*\sigma_0$. It follows from the f-invariance of σ_s that $\sigma_s(z)$ is independent of the choice of (ϕ, U).
2. Complex analyticity of the maps $(s,z)\mapsto f_s(z)$ and $(s,z)\mapsto\phi_s(z)$ follow immediately from Theorem 2.7 applied to the following two commutative diagrams:

$$\begin{array}{ccc} & \xrightarrow{h_s} & \\ f\big\downarrow & & \big\downarrow f_s \\ & \xrightarrow{h_s} & \end{array}\ , \qquad \begin{array}{ccc} & \xrightarrow{h_s} & \\ \phi\big\downarrow & & \big\downarrow \phi_s \\ & \xrightarrow{h_s} & \end{array}\ .$$

3. As h_s is a homeomorphism and $f_s(z)=h_s\circ f\circ h_s^{-1}$ it follows that $\alpha_s=h_s(\alpha)$ is a k-periodic point. Moreover

$$\phi_s\circ f_s^k\circ\phi_s^{-1}=l_s\circ\phi\circ h_s^{-1}\circ h_s\circ f^k\circ h_s^{-1}\circ h_s\circ\phi^{-1}\circ l_s^{-1}=l_s\circ\phi\circ f^k\circ\phi^{-1}\circ l_s^{-1}.$$

   If $\lambda\neq 0$ then $\phi\circ f^k\circ\phi^{-1}(z)=\lambda z$ and

$$\phi_s\circ f_s^k\circ\phi_s^{-1}(z)=l_s\circ\lambda\cdot\circ l_s^{-1}(z)=\lambda_s z=\lambda|\lambda|^{s-1}z.$$

   If $\lambda=0$, we have $f^k(z)-\alpha=a(z-\alpha)^d+\text{higher order terms}$ for some $a\neq 0$ and $\phi\circ f^k\circ\phi^{-1}(d)=z^d$ then

$$\phi_s\circ f_s^k\circ\phi_s^{-1}(z)=l_s\circ z^d\circ l_s^{-1}(z)=z^d.$$

   Moreover $f_s^k(z)-\alpha_s=a(s)(z-\alpha_s)^d+\text{higher order terms}$ with $a(s)=(\phi'(\alpha_s))^{d-1}$ depending holomorphically on s.
   The facts that $\widetilde{B}(\alpha_s)=h_s(\widetilde{B}(\alpha))$ and $B(\alpha_s)=h_s(B(\alpha))$ are immediate from the definition.
4. The extension of ϕ_s is immediate from the definitions.
5. The map $(s,z)\mapsto h_s$ is a holomorphic motion by the measurable Riemann mapping theorem with parameters. Each h_s is automatically a dynamical conjugacy by

the commutative diagram. To prove the group action properties, just look at the following diagram:

$$\begin{array}{ccc}
f,\ W \supset \widetilde{B}(\alpha_f) \supset U & \stackrel{\phi=\phi_f}{\longrightarrow} & \phi_f(U) \subset \mathbb{C}, \\
\Big\downarrow h_{s,\,f} & & \Big\downarrow l_s \\
(s,\, f) = f_s,\ W_s \supset \widetilde{B}(\alpha_s) \supset U_s & \stackrel{\phi_s := l_s \phi h_s^{-1}}{\longrightarrow} & \phi_s(U_s) \subset \mathbb{C} \\
\Big\downarrow h_{s',\, f_s} & & \Big\downarrow l_{s'} \\
g = (s',\, f_s),\ (W_s)_{s'} \supset \widetilde{B}(\alpha_g) \supset (U_s)_{s'} & \stackrel{\phi_g := l_{s'} \phi_s h^{-1}_{s',\, f_s}}{\longrightarrow} & \phi_g((U_s)_{s'}) \subset \mathbb{C}.
\end{array}$$

As $l_{s'} \circ l_s = l_{s' \star s}$, the map $h_{s',\, f_s} \circ h_{s,\, f}$ integrates the complex structure pulled back by $l_{s' \star s} \circ \phi_f$. So, up to normalization, $h_{s',\, f_s} \circ h_{s,\, f} = h_{s' \star s,\, f}$ and $(f_s)_{s'} = f_{s' \star s}$. Similarly

$$\phi_g = l_{s'} \circ \phi_s \circ h^{-1}_{s',\, f_s} = l_{s'} \circ l_s \circ \phi \circ h_s^{-1} \circ h^{-1}_{s',\, f_s} = l_{s' \star s} \circ \phi \circ h^{-1}_{s' \star s}.$$

6. **Injectivity.** If $s_0 = 1$ so that $g = M \circ f \circ M^{-1}$. Then $M^*(\sigma_{s,\, g}) = \sigma_{s,\, f}$ so that $M_s = h_{s,\, g} \circ M \circ h^{-1}_{s,\, f}$ is a Möbius transformation, depends holomorphically on $s \in \mathbb{L}$ and conjugates $(f_s,\, W_{f_s},\, \alpha_{f_s})$ and $(g_s,\, W_{g_s},\, \alpha_{g_s})$. For the general case note that by the group action property for any $s \in \mathbb{L}$ the inverse map $h^{-1}_{s,\, f} : \bar{\mathbb{C}} \longrightarrow \bar{\mathbb{C}}$ integrates the f_s invariant almost complex structure $\sigma_{s^{-1},\, f_s}$, i.e., $(h^{-1}_{s,\, f})^*(\sigma_0) = \sigma_{s^{-1},\, f_s}$. And simillarly for g. If $(f_{s_0},\, W_{f_{s_0}},\, \alpha_{f_{s_0}})$ and $(g_{s_0},\, W_{g_{s_0}},\, \alpha_{g_{s_0}})$ are Möbius conjugate by some M. Then $M^*(\sigma_{s_0^{-1},\, g_{s_0}}) = \sigma_{s_0^{-1},\, f_{s_0}}$ and hence $M_1 = h^{-1}_{s_0,\, g} \circ M \circ h_{s_0,\, f}$ is a Möbius conjugacy between $(f,\, W_f,\, \alpha_f)$ and $(g,\, W_g,\, \alpha_g)$. So the first part applies and yields $M_{s_0} = M$.

□

$$\begin{array}{ccc}
\underset{f\circlearrowright}{\mathbb{C}}\ (\sigma_{s,f}, \sigma_0) & \stackrel{h_{s,f}}{\longrightarrow} & \underset{f_s\circlearrowright}{\mathbb{C}}\ (\sigma_0, \sigma_{s^{-1}, f_s}) \\
\downarrow M_1 & & \downarrow M_s \\
\underset{g\circlearrowright}{\mathbb{C}}\ (\sigma_{s,g}, \sigma_0) & \stackrel{h_{s,g}}{\longrightarrow} & \underset{g_s\circlearrowright}{\mathbb{C}}\ (\sigma_0, \sigma_{s^{-1}, g_s}) \\
\downarrow \phi_g & & \downarrow \phi_{g_s} \\
\mathbb{C}(l_s^*(\sigma_0), \sigma_0) & & \mathbb{C}(\sigma_0,\, l^*_{s-1}(\sigma_0))
\end{array}.$$

3. Centers, stabilizers and proper attracting dynamics. In this section we study the mapping properties of $s \mapsto f_s$ in a BH-motion. We will show that it is either locally injective (as in Example 1, page 5) or constant (rigid, as in Example 2, Page 5). For this we will need some notations:

Definition 3.1. In a BH-motion of an attracting dynamics (f, W, α), we denote by $\mathrm{Stab}(f) \subset \mathbb{L}$, the stabilizer, to be the set of s for which (f_s, W_s, α_s) is Möbius conjugate to (f, W, α).

Definition 3.2. We say that an attracting dynamics (f, W, α) is **proper**, if on every connected component Ω of $\widetilde{B}(\alpha)$ the restriction $f : \Omega \longrightarrow f(\Omega)$ is a proper map (for example a rational map with an attracting cycle and with the choice $W = \bar{\mathbb{C}}$ is always proper).

We shall use the term **central orbit** synonymously with the grand orbit of α : $\mathrm{G.O.}(\alpha) = \{z | \exists n \in \mathbb{N} : f^n(z) = \alpha\}$.

A proper attracting dynamics (f, α) is a **center** if all critical points in $\widetilde{B}(\alpha)$ are central, i.e., belongs to the central orbit, in particular α is f-superattracting. In other words, f is not a center if either $\lambda(f) \neq 0$ or $\lambda(f) = 0$, but at least one critical point in $\widetilde{B}(\alpha)$ is not a preimage of α.

Theorem 3.3 (Injective or rigid)**.** Let (f, W, α) be a proper attracting dynamics with $k(f) = k$, $\lambda(f) = \lambda$.

- *Assume that (f, α) is not a center. Then* $\mathrm{Stab}(f)$ *is a discrete subgroup of* $\mathbb{W} := (1 + i\mathbb{R}, \star)$ *and the* map $s \mapsto f_s$ *is injective on the semi strips* $\{\Re(s) > 0, |\Im(s - s_0)| < \delta\}$ *for some $\delta > 0$ and any $s_0 \in \mathbb{L}$. Moreover* $\mathrm{Stab}(f_s) = s^{-1} \star \mathrm{Stab}(f) \star s$.
- *Assume now (f, α) IS a center. Then* $\mathrm{Stab}(f) \equiv \mathrm{Stab}(f_s) = \mathbb{L}$ *and $f_s \equiv f$ for all $s \in \mathbb{L}$ (after suitable normalization of h_s).*

Remark that Branner and Hubbard's original result ([BH]'s Proposition 8.3, see also [Wi]'s Proposition 5.5) corresponds to the case that f is a polynomial and (f, ∞) is a center, which means in this case the absence of escaping critical points, or equivalently the connectedness of the Julia set.

The first step in our proof is the following:

Proposition 3.4. *For any attracting dynamics (f, α) (with $k(f) = k$, $\lambda(f) = \lambda$), the stabilizer* $\mathrm{Stab}(f)$ *is a subgroup of* $(\mathbb{L}, \star)$ *and is independent of the normalizations of h_s. If $\lambda \in \mathbb{D}^*$ then* $\mathrm{Stab}(f)$ *is a discrete subgroup of* $\mathbb{W}$*. If $\lambda = 0$ and if* $\mathrm{Stab}(f)$ *is not discrete, then there is a sequence $s'_n \overset{\neq}{\longrightarrow} 1$ such that $f_{s'_n} \equiv f$, after suitable normalization of h_s. Consequently $f_s \equiv f$ under this normalization.*

Proof. Assume $s_1, s_2 \in \mathrm{Stab}(f)$. We shall prove that $s_1^{-1} \star s_2 \in \mathrm{Stab}(f)$, which implies that $(\mathrm{Stab}(f), \star)$ is a group.

By Theorem 2.5 5. the maps $s \mapsto f_{s\star s_1}$, $s \mapsto f_{s \star s_2}$ are BH-motions of f_{s_1} and f_{s_2} respectively.

For $i = 1, 2$ let M_i be Möbius transformations with $M_i(\alpha) = \alpha_{s_i}$ and $f = M_i^{-1} \circ f_{s_i} \circ M_i$. Then $f_{s_1} = N^{-1} \circ f_{s_2} \circ N$, where $N = M_2 \circ M_1^{-1}$. By Theorem 2.5 6. the dynamics $f_{s\star s_1}$ and $f_{s \star s_2}$ are Möbius conjugate for all $s \in \mathbb{L}$. Setting $s = s_1^{-1}$ we get that f and $f_{s_1^{-1} \star s_2}$ are Möbius conjugate, i.e., $s_1^{-1} \star s_2 \in \mathrm{Stab}(f)$.

Therefore Stab(f) is a subgroup. Now different normalizations of h_s lead to Möbius conjugated f_s, and therefore the same Stab(f).

Assume now $\lambda \in \mathbb{D}^*$. A necessary condition for $s \in \mathrm{Stab}(f)$ is that $\lambda_s = \lambda$, i.e., $s \in \{1 + i\frac{2\pi n}{\log|\lambda|} \,|n \in \mathbb{Z}\}$ which is a discrete subgroup of $\mathbb{W}$.

Assume now $\lambda = 0$. We will make a sequence of Möbius conjugations to reduce f to a suitable normal form.

We start by remarking that there is a Möbius transformation G with $G(\alpha) = 0$ such that in a neighborhood of the origin

$$G^{-1} \circ f^k \circ G(z) = z^d(1 + \mathcal{D}(z)) = 1 \cdot z^d + p \cdot z^{d+1} + \mathfrak{O}(z^{d+2}),$$

where d is the local degree of f^k at α.

Next we choose a further Möbius transformation M fixing 0 so that $(G \circ M)^{-1} \circ f^k \circ G \circ M(z)$ has a local expansion in the following normal form

$$z^d(1 + \mathcal{D}(z^2)) = 1 \cdot z^d + 0 \cdot z^{d+1} + \mathfrak{O}(z^{d+2}). \tag{3}$$

By looking at the local expansions one can check easily that such M exists, and there are precisely $d - 1$ of them, in the form

$$M(z) = N(\rho z), \;\; \rho^{d-1} = 1, \;\; N(z) = \frac{z}{\frac{p}{d}z + 1}. \tag{4}$$

It follows easily that there are exactly $d - 1$ choices of the composed Möbius map $G \circ M$ to reduce f to its normal form.

Now for each s we may and shall post-compose h_s by Möbius maps $G_s \circ M_s$ to reduce f_s to its normal form. Again there are exactly $d - 1$ choices for each given s. By Theorem 2.5 2. the map $s \mapsto f_s$ is analytic. So $G_s \circ M_s$ can be chosen to be s-analytic.

Therefore there are s-analytic normalizations of h_s so that all f_s have the above normal form. We may and shall thus suppose that all f_s have already the above normal form.

Assume that Stab(f) is not discrete. Then there is a sequence $s_n \in \mathrm{Stab}(f)$ with $s_n \xrightarrow{\neq} s' \in \mathbb{L}$. Then there is a sequence of Möbius transformations M_n with $M_n(0) = 0$ and $M_n^{-1} \circ f \circ M_n = f_{s_n}$. As both f and f_{s_n} have the normal form (3), by (4) we conclude that $M_n(z) = \rho_n z$ with $\rho_n^{d-1} = 1$.

If there is a subsequence (n_p) for which $\rho_{n_p} = 1$, then $f_1 = f_{s_{n_p}}$. But the right hand side converges to $f_{s'}$. So $f_{s_{n_p}} = f_1 = f_{s'}$.

Otherwise there is a subsequence with $\rho_{n_p} = \rho$ for some fixed ρ with $\rho^{d-1} = 1$. Thus $M_{n_p}(z) = M(z) = \rho z$ and $M^{-1} \circ f \circ M = f_{s_{n_p}}$. Again the right hand side converges to $f_{s'}$. So $f_{s_{n_p}} = M^{-1} \circ f \circ M = f_{s'}$.

In both cases we get $f = (f_{s'})_{(s')^{-1}} = f_{(s')^{-1} \star s_{n_p}}$. Setting $s_n' = (s')^{-1} \star s_{n_p}$, we get the proposition, except the final consequence.

But due to Theorem 2.5 2. the map $s \mapsto f_s(z)$ is analytic for each fixed z. By the isolated zero theorem we conclude that $f_s(z) \equiv f(z)$. $\square$

Using Riemann-Hurwitz formula, it is quite easy to prove that (f, α) is a center if and only if any connected component Λ of $\widetilde{B}(\alpha)$ is simply connected and has a unique point in the central orbit.

Proof of Theorem 3.3, non-center part. If $\lambda \in D^\star$ we know already that $\mathrm{Stab}(f) \subset \{1 + i\frac{2\pi n}{\log|\lambda|}\, | n \in \mathbb{Z}\}$ which is a discrete subgroup of $\mathbb{W}$.

For $\lambda = 0$ we have to work a little harder. Let $\phi : U \longrightarrow V$ be a Böttcher coordinate for f^k near α.

Case 1. There is at least one non central critical point in the immediate basin $B(\alpha)$. There is a maximal radius $0 < r < 1$ and an open subset $U^0 \subset B(\alpha)$ such that ∂U^0 contains at least one and at most finitely many critical points c^j and ϕ extends as a biholomorphic map $\phi : U^0 \longrightarrow \mathbb{D}(\mathrm{r})$. The radius r is a conformal invariant. For the Böttcher coordinate ϕ_s of f_s the maximal radius $r(s) = r^{\Re(s)}$. Hence if $\Re(s) \neq 1$ then f_1 and f_s can not be Möbius conjugate, yielding $\mathrm{Stab}(f) \subset \mathbb{W} := \{s, \Re(s) = 1\}$.

Assume that $\mathrm{Stab}(f)$ is not discrete. Then by Proposition 3.4 we have $f_s \equiv f$ after suitable normalization of h_s. But $r(s) = r^{\Re(s)}$ is not constant, which is a contradiction.

Case 2. The point α is the sole critical point of f^k in $B(\alpha)$. Then ϕ extends to a biholomorphic map $\phi : B(\alpha) \longrightarrow \mathbb{D}$ and by assumption there is at least one connected component Ω of $\widetilde{B}(\alpha)$ containing a critical point not in the central orbit. Let n be the minimal iterate for which $f^n(\Omega) = B(\alpha)$ and let r be the maximal modulus of the critical values of $\phi \circ f^n$ on Ω. Then again r is a conformal invariant, and the corresponding value for f_s is $r(s) = r^{\Re(s)}$. Reasoning as above we may conclude that $\mathrm{Stab}(f)$ is again a discrete subgroup of $\mathbb{W}$.

Finally we show that $s \mapsto f_s$ is injective on the semi strips. Let $\delta > 0$ be minimal so that $1 + \delta i \in \mathrm{Stab}(f)$. Then $\mathrm{Stab}(f) = \{1 + in\delta, n \in \mathbb{Z}\}$ and

$$
\begin{aligned}
f_{s_1} = f_{s_2} &\Longrightarrow s_2^{-1} \star s_1 \in \mathrm{Stab}(f) \Longrightarrow \exists n \in \mathbb{Z},\;\; s_1 = s_2 \star (1 + in\delta) \\
&\Longrightarrow \Im(s_1 - s_2) \in \delta\mathbb{Z}.
\end{aligned}
$$

$\square$

Rather than proving now the center part, we prove at first a slightly more general result, unrelated to BH-motions:

Definition 3.5. We say that two attracting dynamics (f, W, α) and (f_0, W_0, α_0) are **hybridly equivalent**, if there is a q.c. homeomorphism $h : \bar{\mathbb{C}} \longrightarrow \bar{\mathbb{C}}$, $h(\alpha) = \alpha_0$, $h(W) = W_0$ which is conformal a.e. on $\widetilde{B}(\alpha)^c$, and which is a conjugacy $h \circ f = f_0 \circ h$ on a neighborhood of $\widetilde{B}(\alpha)^c$. We also call h a **hybrid conjugacy**.

Proposition 3.6 (rigidity). *Let (f_1, W_1, α_1) and (f_2, W_2, α_2) be two proper attracting dynamics which are centers and which are hybridly equivalent to each other by a quasi-conformal map h (in particular $k(f_1) = k(f_2) := k$). Then they are Möbius conjugate (see Definition* 2.6*), by a Möbius transformation M, which coincides with h on $\widetilde{B}(\alpha)^c$.*

Proof. Let $\rho_i : B(\alpha_i) \longrightarrow \mathbb{D}$ denote Riemann maps (Böttcher coordinates) such that $\rho_i \circ f_i^k = (\rho_i)^{p+1}$, where $p+1 = \deg(f^k : B(\alpha) \to B(\alpha))$. The maps are unique modulo multiplication by a pth-root of unity. Let $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a hybrid conjugacy. Then the quasi-conformal homeomorphism $\eta := \rho_2 \circ h \circ \rho_1^{-1} : \mathbb{D} \longrightarrow \mathbb{D}$ extends by reflection to a global quasi-conformal homeomorphism and conjugates $z \mapsto z^{p+1}$ to itself on a neighborhood of $\mathbb{S}^1$. It follows that its restriction to $\mathbb{S}^1$ equals to a rigid rotation of order p, that is ωz with $\omega^p = 1$ (see Lemma 3.8 below and its trailing remark). Hence given a choice of ρ_1 we can choose ρ_2 such that the restriction of η to $\mathbb{S}^1$ is the identity. Define $\phi = \rho_2^-1 \circ \rho_1 : \mathrm{B}(\alpha_1) \longrightarrow \mathrm{B}(\alpha_2)$. We shall also express this fact that $\eta(z) = z$ on $\mathbb{S}^1$ by saying that ϕ and h are identical on the ideal boundary or $h^{-1} \circ \phi$ is the identity on the ideal boundary. See the following diagram.

$$
\begin{array}{ccccc}
\Omega & \xrightarrow{f_1^n} & B(\alpha_1) & \xrightarrow{\rho_1} & \mathbb{D} \\
h \downarrow\downarrow \phi_\Omega & & h \downarrow\downarrow \phi & & \eta \downarrow\downarrow id \\
h(\Omega) & \xrightarrow[f_2^n]{} & B(\alpha_2) & \xrightarrow[\rho_2]{} & \mathbb{D}
\end{array}
\tag{5}
$$

Since η is quasi-conformal and equal to the identity on $\mathbb{S}^1$, there exists a constant $C > 0$ depending only on the maximal dilatation of η (and thus implicitly on h) such that $\forall z \in \mathbb{D}$: $\mathrm{d}_{\mathbb{D}}(z, \eta(z)) \leq C$ or equivalently $\forall z \in \mathrm{B}(\alpha_1)$: $\mathrm{d}_{\mathrm{B}}(\alpha_2)(h(z), \phi(z)) \leq C$. This is a classic compactness result for q.c. mappings. For completeness we reprove it in Lemma 3.9 below.

For Ω any connected component of $\widetilde{B}(\alpha_i)$ let $n = n(\Omega) = \min\{m \mid f_i^m(\Omega) = B(\alpha_i)\}$ and $p(\Omega) = \deg(f_i^n : \Omega \to B(\alpha_i))$. Then $p(\Omega) = p(h(\Omega))$ and $n(\Omega) = n(h(\Omega))$ and $\phi \circ f_1^n$ lifts by f_2^n to an isomorphism $\phi_\Omega : \Omega \longrightarrow h(\Omega)$. This lift is uniquely determined up to post composition by a deck-transformation for f_2^n. Also h and ϕ_Ω differs on the ideal boundary by a deck-transformation for the action of f_2^n on the ideal boundary. We let ϕ_Ω be the unique choice of lift for which h and ϕ_Ω are identical on the ideal boundary of Ω. Then by the same argument as above we have $\forall z \in \Omega$: $\mathrm{d}_{h(\Omega)}(h(z), \phi_\Omega(z)) \leq C$, with the same C, since $h \circ \phi_\Omega^{-1}$ is K-qc with the same K as above.

Define $H : \bar{\mathbb{C}} \longrightarrow \bar{\mathbb{C}}$ by $H = h$ on $\widetilde{B}(\alpha_1)^c$ and by $H = \phi_\Omega$ on each connected component Ω of $\widetilde{B}(\alpha_1)$.

In order to prove that H is continuous, choose two distinct points β, γ outside $\widetilde{B}(\alpha_1)$. Let $w_n \in \Omega_n$, where the Ω_n are (not necessarily distinct) connected components of

$\widetilde{B}(\alpha_1)$ and $w_n \to w \in \partial\widetilde{B}(\alpha_1)$. (The point w may be one of β, γ, but is never $\alpha_1 \in \widetilde{B}(\alpha_1)$). Then

$$\mathrm{d}_{\bar{\mathbb{C}}\setminus h(\{\alpha_1,\beta,\gamma,w\})}(h(w_n), H(w_n)) \le d_{h(\Omega_n)}(h(w_n), \phi_{\Omega_n}(w_n)) \le C < \infty.$$

By a classical inequality (see for example Milnor [Mi]) we have $H(w_n) \to h(w)$ as $n \to \infty$.

Thus H is a homeomorphism, which coincides with h outside $\widetilde{B}(\alpha_1)$.

But then by Rickman's lemma ([Ri], see also [DH2], Lemma 2) H is also quasi-conformal, because h is globally quasi-conformal and the patches ϕ_Ω are also quasi-conformal, in fact conformal. Moreover H is 1-quasi-conformal, because the maps ϕ_Ω are conformal and h is conformal a.e. on $\widetilde{B}(\alpha_1)^c$. Finally H is conformal and thus a Möbius transformation by Weyl' s lemma. □

Remark. Note that the key point here is the existence of bi-holomorphic conjugacies ϕ, ϕ_Λ equal to h on the ideal boundaries, as indicated in the diagram (5). We may thus replace the assumption of being centers by this requirement and obtain a more general rigidity result. We will need this fact twice in Section 4.

Definition 3.7. A degree $d \ge 2$ orientation preserving covering map f: $\mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is called weakly expanding iff $\forall x, y \in \mathbb{S}^1$ and for each of the two complementary subarcs $I_1 = [x, y]$ and $I_2 = [y, x]$ of $\{x, y\}$ in $\mathbb{S}^1$ there exists an $n \in \mathbb{N}$ such that $f^n(I_i) = \mathbb{S}^1$ or equivalently f^n is not injective on any of the two arcs.

The following Lemma is a classical result included for completeness.

Lemma 3.8. *For any pair of degree $d \ge 2$ weakly expanding covering* maps $f_i : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ *and any choice of fixed points $\alpha_i \in \mathbb{S}^1$, $f_i(\alpha_i) = \alpha_i$ for $i = 1, 2$. There exists a unique orientation preserving homeomorphism $h : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ with $h(\alpha_1) = \alpha_2$ and $h \circ f_1 = f_2 \circ h$.*

Proof. Existence: Let $h_0 : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be any orientation preserving homeomorphism with $h_0(\alpha_1) = \alpha_2$, e.g. $h_0(z) = \frac{\alpha_1}{\alpha_2} z$. Define recursively $h_n : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ to be the unique lift of $h_{n-1} \circ f_1$ to f_2, with $h_n(\alpha_1) = \alpha_2$. (Equivalently define h_n to be the unique lift of $h_0 \circ f_1^n$ to f_2^n.) Then each h_n is order preserving and for every $m \ge n : h_m(f_1^{-n}(\alpha_1)) = f_2^{-n}(\alpha_2)$. As each f_i is weakly expanding, both families $\{h_n\}_n$ and $\{h_n^{-1}\}_n$ are equicontinuous and hence pre-compact. Let $h : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be any limit map. Then h is a homeomorphism and $h \circ f_1 = f_2 \circ h$ on the subset, $\cup_n f_1^{-n}(\alpha_1)$, which is dense because f_1 is weakly expanding. Hence $h \circ f_1 = f_2 \circ h$ on $\mathbb{S}^1$ as desired.

Uniqueness: If $\widehat{h} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is any orientation preserving conjugacy with $\widehat{h}(\alpha_1) = \alpha_2$, then $\widehat{h} = h$ on the dense subset $\cup_n f_1^{-n}(\alpha_1)$ and hence everywhere in $\mathbb{S}^1$. □

As an immediate consequence of this Lemma the automorphism group of $z \mapsto z^d$ for $d \ge 2$ (i.e., the set of orientation preserving homeomorphisms which commutes with z^d) equals the set of rigid rotations $\{z \mapsto \rho z | \rho^{d-1} = 1\}$, since $\{\rho | \rho^{d-1} = 1\}$ equals the set of fixed points in $\mathbb{S}^1$ of z^d. $\square$

Lemma 3.9. *There exists* $C = C(K) > 0$ *such that for any* K*-qc homeomorphism,* $h : \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$ *with* $h = \mathrm{id}$ *on* $\mathbb{S}^1$: $\forall z \in \mathbb{D} : \mathrm{d}_{\mathbb{D}}(z, h(z)) \leq C$.

Proof. Define

$$\mathcal{K}_K = \{h : \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}} | h \text{ is } K\text{−}qc \text{ and } h = \mathrm{id} \text{ on } \mathbb{S}^1\}.$$

Then $\mathcal{K}_K$ is compact, because any $h \in \mathcal{K}_K$ extends by Schwarz-reflection in $\mathbb{S}^1$ to a global K-qc map, which fixes three distinct points say $1, i, -1$. Since the map $h \mapsto \mathrm{d}_{\mathbb{D}}(0, h(0))$ is continuous on the compact set $\mathcal{K}_K$ we can define $C = C(K)$ as its maximal value.

Let $h \in \mathcal{K}_K$ and $z_0 \in \mathbb{D}$ be arbitrary and let $M(z) = \frac{z + z_0}{1 + \bar{z}_0 z}$, so that $M(0) = z_0$. Then $M^{-1} \circ h \circ M \in \mathcal{K}_K$ and since M is a hyperbolic isometry we have

$$\mathrm{d}_{\mathbb{D}}(z_0, h(z_0)) = \mathrm{d}_{\mathbb{D}}(0, M^{-1} \circ h \circ M(0)) \leq C.$$

$\square$

End of the proof of Theorem 3.3, the center part. This can be deduced easily as follows: We normalize h_s so that it fixes α and two points of $\widetilde{B}(\alpha)^c$. For any s the maps f_1, f_s and h_s satisfy the hypothesis of Proposition 3.6 and the Möbius conjugacy M_s fixes three points on the sphere. So $f_s \equiv f$ and $\mathrm{Stab}(f) = \mathbb{L}$. $\square$

Remark. The proof of Proposition 3.6 can also be done explicitly using the formula of $\widetilde{l}_s$.

4. Applications.

4.1. Cubic slices. The first of our two examples is the two parameter family of cubic polynomials

$$P_{\lambda, a}(z) := \lambda z + \sqrt{a} z^2 + z^3, \qquad \lambda, a \in \mathbb{C}. \tag{6}$$

Here the two different determinations of $\sqrt{a}$ yields maps which are conjugate by the map $z \mapsto -z$ and hence are holomorphically equivalent. Moreover any cubic polynomial admitting 0 as a fixed point is linearly conjugate to $P_{\lambda, a}$ for some unique parameters λ, a. Let

$$\widetilde{\mathcal{H}} = \{(\lambda, a) \in \mathbb{D} \times \mathbb{C} \mid \text{both critical points belong to } \widetilde{B}_{\lambda, a}(0)\}$$

$$\widetilde{\mathcal{H}}^c = \{(\lambda, a) \in \mathbb{D} \times \mathbb{C} \mid \text{only one simple critical point belongs to } \widetilde{B}_{\lambda, a}(0)\}$$

$$\mathcal{P}_\lambda = \{(\lambda', a) | \lambda' = \lambda\} \simeq \mathbb{C}, \qquad \widetilde{\mathcal{H}}_\lambda = \widetilde{\mathcal{H}} \cap \mathcal{P}_\lambda, \qquad \text{in particular}$$

$$\widetilde{\mathcal{H}}^c_{e^{-1}} = \{a \in \mathbb{C} \mid \text{only one simple critical point belongs to } \widetilde{B}_{e^{-1}, a}(0)\}.$$

We study the effect of the BH-motion on both the parameter space and the dynamical plane, and determine completely the stabilizers.

Theorem 4.1. *There exists a holomorphic motion* $H : (s, a, z) \mapsto (v(s, a), h(s, a, z))$, $\mathbb{L} \times \mathbb{C}^2 \to \mathbb{C}^2$ *over* $\mathbb{L}$ *based at* $s_0 = 1$ *such that for* $\lambda_0 = \mathrm{e}^{-1} = \mathrm{e}^{-s_0}$*:*

1. *For each fixed a, the map* $s \mapsto h(s, a, \cdot)$ *is a BH-motion of the attracting dynamics* $(P_{e^{-1}, a}, \bar{\mathbb{C}}, 0)$*, in particular* $h_{s, a} \circ P_{e^{-1}, a} = P_{e^{-s}, v(s, a)} \circ h_{s, a}$.
2. *For each fixed* (s, a) *the quasi-conformal conjugacy* $z \mapsto h(s, a, z)$ *is conformal on the exterior of* $\widetilde{B}_{e^{-1}, a}(0)$.
3. *The holomorphic motion* $(s, a) \mapsto v(s, a)$ *restricted to* $\mathbb{L} \times \widetilde{\mathcal{H}}^c_{e^{-1}}$ *is* $2\pi i$*-periodic in the s variable. In particular* $\mathrm{Stab}(P_{e^{-1}, a}) = 1 + 2\pi i\mathbb{Z}$*. Furthermore the holomorphic motion H restricted to* $\{(s, a, z), s \in \mathbb{L}, a \in \widetilde{\mathcal{H}}^c_{e^{-1}}, z \in \widetilde{B}_{e^{-1}, a}(0)^c\}$ *is also* $2\pi i$*-periodic in the s variable* (*but has a more complicated monodromy structure elsewhere*).

By the λ-lemma for holomorphic motions, the maps $(s, z) \mapsto h(s, a, z)$ for $a \in \mathbb{C}$ fixed and $(s, a) \mapsto v(s, a)$ are continuous as functions of two variables. However in general the map $h(s, a, z)$ is discontinuous with respect to a.

The periodicity leads naturally to the following operations:

Define $\widehat{v} : \mathbb{D}^\star \times \widetilde{\mathcal{H}}^c_{e-1} \longrightarrow \mathbb{C}$ by $\widehat{v}(e^{-s}, a) = v(s, a)$; define $\widehat{h} : \mathbb{D}^\star \times \widetilde{\mathcal{H}}^c_{e-1} \times \widetilde{B}_{e^{-1}, a}(0)^c \longrightarrow \mathbb{C}$ by $\widehat{h}(e^{-s}, a, z) = h(s, a, z)$, and finally define

$$\widehat{H} : \{(\lambda, a, z), \lambda \in \mathbb{D}^\star, a \in \widetilde{\mathcal{H}}^c_{e-1}, z \in \widetilde{B}_{e^{-1}, a}(0)^c\} \longrightarrow \bar{\mathbb{C}}^2$$

by $\widehat{H}(\lambda, a, z) = (\widehat{v}(\lambda, a), \widehat{h}(\lambda, a, z))$. Then both $\widehat{v}(\cdot, \cdot)$ and $\widehat{H}(\cdot, a, \cdot)$ are holomorphic motions over $\mathbb{D}^*$ with base point $\lambda_0 = \mathrm{e}^{-1}$. We call them the quotient motions.

With a little extra work we obtain:

Theorem 4.2. *The quotient holomorphic motions* $\widehat{v}(\cdot, \cdot)$ *and* $\widehat{H}(\cdot, a, \cdot)$ *both extend to holomorphic motions over* $\mathbb{D}$ *with base point* 0.

Proof of Theorem 4.1. For $a \in \mathbb{C}$ fixed and a fixed choice of $\sqrt{a}$, we consider a BH-motion for the attracting dynamics $(f, W, \alpha) := (P_{e^{-1}, a}, \bar{\mathbb{C}}, 0)$. We normalize the integrating maps $h_s = h_{s,a}$ so that $h_{s,a}(0) = 0$, $h_{s,a}(\infty) = \infty$ and $h_{s,a}$ is tangent to the identity at ∞. This implies that the new maps f_s are again 2 cubic polynomials in the form (6). Further, by Theorem 2.5 3. the new multipliers $\lambda(f_s)$ equal to $e^{-1}|e^{-1}|^{s-1} = e^{-s}$. Therefore $f_s = P_{e^{-s}, v(s,a)}$ for some $v(s, a) \in \mathbb{C}$. Set $h(s, a, z) := h_{s,a}(z)$ and

$$H(s, a, z) := (v(s, a), h(s, a, z)).$$

We check at first that $H : \mathbb{L} \times \mathbb{C}^2 \to \mathbb{C}^2$ is a holomorphic motion:

- Injectivity on (a, z): For any fixed $s \in \mathbb{L}$, assume $H(s, a, z) = H(s, a', z')$. In particular $v(s, a) = v(s, a')$ and so $P_{e^{-s}, v(s, a)} = P_{e^{-s}, v(s, a')}$. By Theorem 2.5 6. we conclude that $P_{e^{-1},a}$ and $P_{e^{-1},a'}$ are linearly conjugate and hence $a = a'$. Consequently $h(s, a, z) = h(s, a, z')$. But $h(s, a, \cdot) = h_{s,a}(\cdot)$ is a homeomorphism of $\mathbb{C}$, so we conclude that $z = z'$.

• Analyticity in s: The map $h(s, a, z)$ is analytic in s for any fixed $a, z \in \mathbb{C}$, by the measurable Riemann mapping theorem with parameters. Due to Theorem 2.5 2. for $a \in \mathbb{C}$ fixed, the map

$$\begin{aligned} s \mapsto P_{e^{-s}, v(s,a)}(z) &= \lambda_s z + \widetilde{v}(s, a) z^2 + z^3 \\ &= h_{s,a} \circ (z \mapsto e^{-1} z + \sqrt{a} z^2 + z^3) \circ h_{s,a}^{-1}(z), \end{aligned}$$

where $v(s, a) = \widetilde{v}(s, a)^2$ is analytic in s for every fixed z. It follows $\widetilde{v}(s, a)$ and hence $v(s, a)$ depends complex analytically on s.

• Identity at the base point $s = 1$. In this case $h_{1,a}(z) \equiv z$ and consequently $P_{e^{-1}, v(1,a)} := h_{1,a} \circ P_{e^{-1},a} \circ h_{1,a}^{-1} = P_{e^{-1},a}$. So $v(1, a) = a$ and $H(1, a, z) = (a, z)$.

This proves that H is indeed a holomorphic motion over $\mathbb{L}$. We proceed to prove the remaining part of Theorem 4.1.

(1) By construction.

(2) is obvious.

(3) Fix now any $a \in \widetilde{\mathcal{H}}^c_{e^{-1}}$, and set $B(0) = B_{e^{-1},a}(0)$ and $\widetilde{B}(0) = \widetilde{B}_{e^{-1},a}(0)$. Recall that by definition of $\widetilde{\mathcal{H}}^c_{e^{-1}}$, the entire attracted basin $\widetilde{B}(0)$ of 0 for $P_{e^{-1},a}$ contains a unique critical point $c_0 = c_0(a)$. This critical point is in the immediate basin $B(0)$ and has local degree 2.

Let $s \in \mathrm{Stab}(P_{e^{-1},a})$. Thus $P_{e^{-s}, v(a,s)}$ is Möbius conjugate to $P_{e^{-1},a}$. A necessary condition on s is that $e^{-s} = e^{-1}$. In other words $\mathrm{Stab}(P_{e^{-1},a}) \subset 1 + 2\pi i\mathbb{Z}$.

Claim. We have $v(1 + 2\pi i, a) = v(1, a) \equiv a$, i.e., $f_{1+2\pi i} = P_{e^{-1}, v(1+2\pi i, a)}$ equals to $f = f_1 := P_{e^{-1},a}$. Furthermore $h_{1+2\pi i, a}(z) = h_{1,a}(z) \equiv z$ for all $z \in \widetilde{B}(0)^c$. Consequently $\mathrm{Stab}(P_{e^{-1},a}) = 1 + 2\pi i\mathbb{Z}$.

Proof of Claim.

Let $\phi : B(0) \longrightarrow C$ be the linearizer with $\phi(c_0) = 1$ and let $\psi : \mathbb{D} \longrightarrow U$ be the local inverse carrying 0 to 0. Define $\widehat{h} := \psi \circ l_{1+2\pi i} \circ \phi : U \longrightarrow U$ so that $\widehat{h}(f(c_0)) = f(c_0)$, and extend $\widehat{h}$ by iterated lifting to a quasi conformal homeomorphism $\widehat{h} : B(0) \longrightarrow B(0)$, which integrates $\sigma_{1+2\pi i}$ and which conjugates f to itself. By further iterated lifting we can uniquely extend $\widehat{h}$ to a q.c. homemorphism $\widehat{h} : \widetilde{B}(0) \longrightarrow \widetilde{B}(0)$ which preserves each connected component of $\widetilde{B}(0)$.

We now prove that $\widehat{h}$ equals to the identity on the ideal boundary of $B(0)$. Let $\eta : B(0) \longrightarrow \mathbb{D}$ denote a Riemann map fixing 0 and define $R = \eta \circ f \circ \eta^{-1}$. Then R is a quadratic Blaschke product fixing the origin. Define $\widetilde{h} = \eta \circ \widehat{h} \circ \eta^{-1}$ and extend $\widetilde{h}$ to a global q.c. homeomorphism $\widetilde{h} : \bar{\mathbb{C}} \longrightarrow \bar{\mathbb{C}}$ by reflection in the unit circle. Then

$\widetilde{h} \circ R = R \circ \widetilde{h}$ and in particular this holds on the unit circle which is invariant by both R and $\widetilde{h}$. However since the degree of R is 2 there is a unique self homeomorphism of $\mathbb{S}^1$, which commutes with R. It is the identity. Thus $\widetilde{h}$ equals to the identity on $\mathbb{S}^1$. It follows that $\widehat{h}$ equals to the identity on the ideal boundary of $B(0)$.

Similarly $\widehat{h}$ equals to the identity on the ideal boundary of every connected component of $\widetilde{B}(0)$. As in the proof of Proposition 3.6 (see also the Remark following its proof), we conclude that the extension

$$\widehat{h} = \begin{cases} \widehat{h} & \text{on} \quad \widetilde{B}(0) \\ id & \text{on} \quad \widetilde{B}(0)^c. \end{cases}$$

is a global quasi-conformal homeomorphism, conjugating f to itself. As $\widehat{h}$ integrates the almost complex structure $\sigma_{1+2\pi i}$, fixes the origin and is tangent to the identity at ∞, it follows that $\widehat{h} = h_{1+2\pi i}$, $f = f_{1+2\pi i}$ and $h_{1+2\pi i}$ equals to the identity on $\widetilde{B}(0)^c$. This ends the proof of the Claim.

Now we may prove that $s \mapsto P_{e^{-s},\, v(s,\, a)}$ is $2\pi i$-periodic, or equivalently $s \mapsto v(s, a)$ is $2\pi i$-periodic. We have

$$v(s + 2\pi i, a) = v(s \star (1 + 2\pi i), a) = v(s, v(1 + 2\pi i, a)) = v(s, a) \tag{7}$$

where the first equality is due to the (simple) equality $s + 2\pi i = s \star (1 + 2\pi i)$, the second is due to the group action property $(f_{s'})_s = f_{s \star s'}$ in Theorem 2.5 5. and the third is due to the claim above.

Assume now $z \in \widetilde{B}(0)^c$, we want to prove that $s \mapsto h(s, a, z)$ is also $2\pi i$-periodic. In particular the maps $h_{1+n \cdot 2\pi i,\, a}(z)$, $n \in \mathbb{Z}$ are the identity on $\widetilde{B}(0)^c$. Again

$$\begin{aligned} h(s + 2\pi i, a, z) &= h(s \star (1 + 2\pi i), a, z) \\ &= h(s, v(1 + 2\pi i, a), h(1 + 2\pi i, a, z)) = h(s, a, z), \end{aligned}$$

where the second equality is due to the group action property $h_{s \star s',\, f} = h_{s,\, f_{s'}} \circ h_{s',\, f}$ in Theorem 2.5 5. and the third is due to the claim above.

In the proof of Theorem 4.2 the hard work is really to prove that $\widehat{H}$ extends to a holomorphic motion to $\lambda = 0$. Because then one can change the base point from e^{-1} to 0 as follows: for any holomorphic motion $K : \Lambda \times E \longrightarrow \chi$ with base point $\lambda_0 \in \Lambda$ define $E_{\lambda_0} := E$ and more generally $E_\lambda := K_\lambda(E)$ for $\lambda \in \Lambda$. Then $K' : \Lambda \times E_{\lambda_1} \longrightarrow \chi$ given by $K'(\lambda, z) = K(\lambda, K_{\lambda_1}^{-1}(z))$ is a holomorphic motion with the same fibers and the same set of graphs $\{K(\Lambda, z) | z \in E_{\lambda_0}\} = \{K'(\Lambda, z) | z \in E_{\lambda_1}\}$, but with base point λ_1.

We start by proving

Lemma 4.3. *Both* maps $\widehat{v} : \mathbb{D}^\star \times \widetilde{\mathcal{H}}^c_{e^{-1}} \longrightarrow \mathbb{C}$ *and* $\widehat{h} : \mathbb{D}^\star \times \widetilde{\mathcal{H}}^c_{e^{-1}} \times \widetilde{B}_{e^{-1},\, a}(0)^c \longrightarrow \mathbb{C}$ *have unique extensions to* $\lambda = 0$ *such that* $\lambda \mapsto \widehat{v}(\lambda, a)$ *and* $\lambda \mapsto \widehat{h}(\lambda, a, z)$ *are holomorphic for every fixed* $a \in \widetilde{\mathcal{H}}^c_{e^{-1}}$ *and every fixed* $z \in \widetilde{B}_{e^{-1},\, a}(0)^c$.

For P a monic polynomial of degree d denote by $g_P : \mathbb{C} \longrightarrow [0, \infty[$ the Böttcher potential at infinity. That is $g = g_P$ is the unique subharmonic function, which satisfies

i) $g(P(z)) = d \cdot g(z)$, ii) $g(z) \equiv 0$ on $\mathbb{C} \backslash B_P(\infty)$ and iii) $g(z) - \log|z| = o(1)$ at ∞. Denote by $g_{\lambda, a}$ the map $g_{P_{\lambda, a}}$.

Proof of Lemma 4.3. The conjugacy $h_{s, a}$ is conformal on the attracted basin of infinity $B_{e^{-1}, a}(\infty)$ for $f = P_{e^{-1}, a}$ and hence preserves the Böttcher potential, i.e., $g_{e^{-s}, v(s, a)}(h(s, a, z)) = g_{e^{-1}, a}(z)$. Hence for any $z \in \widetilde{B}_{e^{-1}, a}(0)^c$ the map $\lambda \mapsto \widehat{h}_a(\lambda, z)$ is bounded. To see this note that the Böttcher coordinate at ∞ for $P_{\lambda, a'}$ is tangent to the identity at ∞ for any λ, $a' \in \mathbb{C}$, and apply the compactness of normalized univalent maps. Similarly $\widehat{v}(\lambda, a)$ is bounded because $\widehat{v}(\lambda, a) \simeq 3c_1(\lambda, a)/2$ for small λ, where $c_1(\lambda, a) = \widehat{h}_a(\lambda, c_1)$ denotes the critical point of $P_{\lambda, \hat{v}(\lambda, a)}$ not in $\widehat{h}_{\lambda, a}(\widetilde{B}_{e^{-1}, a}(0))$ and c_1 denotes the critical point of $P_{e^{-1}, a}$ not in $\widetilde{B}_{e^{-1}, a}(0)$. By the theorem of removable singularities both $s \mapsto \widehat{v}(\cdot, a)$ and $s \mapsto \widehat{h}(\cdot, a, z)$ extends holomorphically to $\lambda = 0$. □

To complete the proof of Theorem 4.2 we need to check that the extended maps $a \mapsto \widehat{v}(0, a)$ and $z \mapsto \widehat{h}_a(0, z)$ are injective on $\widetilde{\mathcal{H}}^c_{e^{-1}}$, respectively on $\widetilde{B}_{e^{-1}, a}(0)^c$ for $a \in \widetilde{\mathcal{H}}^c_{e^{-1}}$. For this we prove at first the following

Theorem 4.4. *For any $a \in \widetilde{\mathcal{H}}^c_{e^{-1}}$ and any $\lambda \in \mathbb{D}$ the map $P_{\lambda, \hat{v}(\lambda, a)}$ is hybridly equivalent to $P_{0, \hat{v}(0, a)}$. More precisely there exist hybrid conjugacies $H_{\lambda, a} : \mathbb{C} \longrightarrow \mathbb{C}$ between $P_{\lambda, \hat{v}(\lambda, a)}$ and $P_{0, \hat{v}(0, a)}$, which are asymptotic to the identity at ∞, such that, for each fixed a, and as $\lambda \to 0$, the dilatations $||\bar{\partial} H_{\lambda, a} / \partial H_{\lambda, a}||_\infty$ converges to zero uniformly, and $H_{\lambda, a}$ converges locally uniformly to the identity.*

Proof. In the following we fix an arbitrary $a \in \widetilde{\mathcal{H}}^c_{e^{-1}}$ and suppress a in the rest of the proof, e.g. write P_λ for $P_{\lambda, \hat{v}(\lambda, a)}$ etc. It is easy to see (see below) that there is a center attracting dynamics (see Definition 3.2) which is hybridly equivalent to $P_{\mathrm{e}^{-1}}$. Once properly normalized, this center is a cubic polynomial of the form $P_{0, b}$, with $b \in \mathbb{C} \backslash \{0\}$ as it is a cubic polynomial with a superattracting fixed point of order 2. Moreover as $\widehat{h}_\lambda$ is a hybrid conjugacy between $P_{\mathrm{e}^{-1}}$ and P_λ, the map $P_{0, b}$ is a common center for all P_λ, $\lambda \in \mathbb{D}$, by uniqueness of centers (Proposition 3.6) . Thus to prove that $\widehat{v}(0, a) = b$ we need only to show that, as $\lambda \to 0$, the coefficients of P_λ converge to those of $P_{0, b}$, or equivalently, the non-captured P_λ-critical point $c_1(\lambda) \in \widehat{B}_{\lambda, \hat{v}(\lambda, a)}(0)^c$ converges to the $P_{0, b}$-critical point $c_1(b) \in \widetilde{B}_{0, b}(0)^c$ (as the captured P_λ-critical point $c_0(\lambda) \in \widehat{B}_{\lambda, \hat{v}(\lambda, a)}(0)$ converges to 0 and the critical points determine P_0) .

To this end let $\phi_\lambda : B_\lambda(0) \longrightarrow \mathbb{D}$ be the Riemann map with $\phi_\lambda(0) = 0$ and $R_\lambda := \phi_\lambda^{-1} \circ P_\lambda \circ \phi_\lambda = z \frac{z+\lambda}{1+\bar{\lambda} z}$. Note that $\overline{\mathbb{D}(|\lambda|)}$ contains the critical point of R_λ in $\mathbb{D}$ as well as both preimages of 0. Define $V'_\lambda := \phi_\lambda^{-1}(\mathbb{D}\,(\sqrt{|\lambda|}))$ and $V_\lambda := P_\lambda^{-1}(V'_\lambda) \cap B_\lambda(0)$ so that $V'_\lambda \subset\subset V_\lambda$ and define a new map $\widehat{P_\lambda} : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$\widehat{P_\lambda} = \begin{cases} P_\lambda & \text{on } \mathbb{C} \backslash V_\lambda \\ \phi_\lambda^{-1} \circ (z \mapsto z^2) \circ \phi_\lambda & \text{on } \overline{V'_\lambda} \\ \text{degree 2 quasi-regular interpolation} & \text{on } A := V_\lambda \backslash \overline{V'_\lambda}. \end{cases}$$

Note that on the boundary of the annulus A the map $\widehat{P_\lambda}$ is a priori defined as real analytic covering maps of degree 2. It easily follows that there exists a, say C^1 extension also denoted $\widehat{P_\lambda}$, $\widehat{P_\lambda} : A \longrightarrow A' := \phi_\lambda^{-1}(\{z||\lambda| \leq |z| \leq \sqrt{|\lambda|}\})$, which is also a degree 2 covering. We need a little more, namely we need that this extension can be chosen so that its complex dilatation converges to 0 as $|\lambda| \to 0$. Using the Riemann map ϕ_λ we can transport the problem to $\mathbb{D}$, solve it and transport the solution back. That it can be solved (for R_λ in $\mathbb{D}$) is a consequence of Lemma 4.5 below. The map $\widehat{P_\lambda}$ is evidently quasi regular. And any point of z passes at most once through the zone A, where $\widehat{P_\lambda}$ is not conformal. With the quasi regular extension on A given by Lemma 4.5 through ϕ_λ let μ denote the measurable $\widehat{P_\lambda}$-invariant Beltrami form, which equals the standard 0 Beltrami form on V'_λ and on $\widetilde{B}_{e^{-1},a}(0)^c$. Let also $H_\lambda : \mathbb{C} \longrightarrow \mathbb{C}$ denote the integrating map for μ, given by the measurable Riemann mapping theorem, and normalized by $H_\lambda(0) = 0$ and H_λ being tangent to the identity at ∞. Then $P_{0,b} = H_\lambda \circ \widehat{P_\lambda} \circ H_\lambda^{-1}$ and the dilatation of H_λ is bounded by that of μ on A, which is bounded by $16 \cdot 2 \cdot \sqrt{|\lambda|}/\log\sqrt{|\lambda|}$, as shown in Lemma 4.5. This bound of distortion tends to 0 as $\lambda \to 0$. With the chosen normalization it follows that H_λ converges uniformly to the identity on compact sets of $\mathbb{C}$. In particular the P_λ-critical point $c_1(\lambda) \in \widetilde{B}_{\lambda,\hat{v}(\lambda,a)}(0)^c$ converges to the $P_{0,b}$-critical point $c_1(b) \in \widetilde{B}_{0,b}(0)^c$. We conclude then $P_{0,b} = P_0$. □

Lemma 4.5. *For each fixed $d > 1$ there exists $0 < r_0 = r_0(d) < 1$ with the following property: Let $R : \mathbb{D} \longrightarrow \mathbb{D}$ be any degree d Blaschke product fixing 0 and 1 for which the critical values and the zeros are contained in $\mathbb{D}(r)$ for some $r < r_0$. Define $\widehat{V}' = \mathbb{D}(\sqrt{r})$ and $\widehat{V} = R^{-1}(\widehat{V}')$ (we have $\widehat{V} \supset \widehat{V}'$ by Schwarz Lemma). Then there exists a degree d quasi regular branched covering $F : \mathbb{D} \longrightarrow \mathbb{D}$ with $F(z) = z^d$ on $\overline{\widehat{V}'}$ and $F(z) = R(z)$ on $\mathbb{D} \backslash \widehat{V}$ such that*

$$\left|\frac{\bar{\partial} F}{\partial F}\right| \leq \frac{16d\sqrt{r}}{|\log\sqrt{r}|}. \tag{8}$$

Proof. Let $A' = \{z|r < |z| < 1\}$, $A = R^{-1}(A')$ and $A'' = \{z|r^{1/d} < |z| < 1\}$. Let $\Gamma : A'' \longrightarrow A$ denote the lift of z^d to R which fixes 1, i.e., $R(\Gamma(z)) = z^d$. Set $C = \{z|r^{1/2d} < |z| < 1\}$. Then it suffices to construct a quasi-conformal map G, which is the identity on $\widehat{V}'$ and equals Γ on C, because then

$$F(z) := (G^{-1}(z))^d$$

would be the required map. Write $r = \exp(dm)$ (with $m < 0$) and let $\widetilde{A}, \widetilde{A}'', \widetilde{C}, \widetilde{V}'$ be the preimages by $\exp(z)$ of the corresponding un-tilded (and hatted for V) sets in $\mathbb{D}^*$. In particular

$$\widetilde{A}'' = \{x + iy|x \in \,]m, 0[\}, \quad \widetilde{C} = \left\{x + iy|x \in \,\left]\frac{m}{2}, 0\right[\right\}, \quad \widetilde{V}' = \left\{x + iy|x < \frac{dm}{2}\right\}.$$

Denote by $\widetilde{\Gamma} : \widetilde{A}'' \longrightarrow \widetilde{A}$ the lift of $\Gamma \circ \exp(z)$ to $\exp(z)$ which fixes 0. Then $\widetilde{\Gamma}(z + 2\pi) = \widetilde{\Gamma}(z) + 2\pi$ and we shall construct a q.c. homeomorphism $\widetilde{G} : \mathbb{H}_- \longrightarrow \mathbb{H}_-$

which equals $\widetilde{\Gamma}$ on $\widetilde{C}$ and the identity on $\widetilde{V'}$, which satisfies $\widetilde{G}(z+2\pi)=\widetilde{G}(z)+2\pi$ as well as (8). Then

$$F(z) := \exp(d\cdot \widetilde{G}^{-1}(\log z)) \tag{9}$$

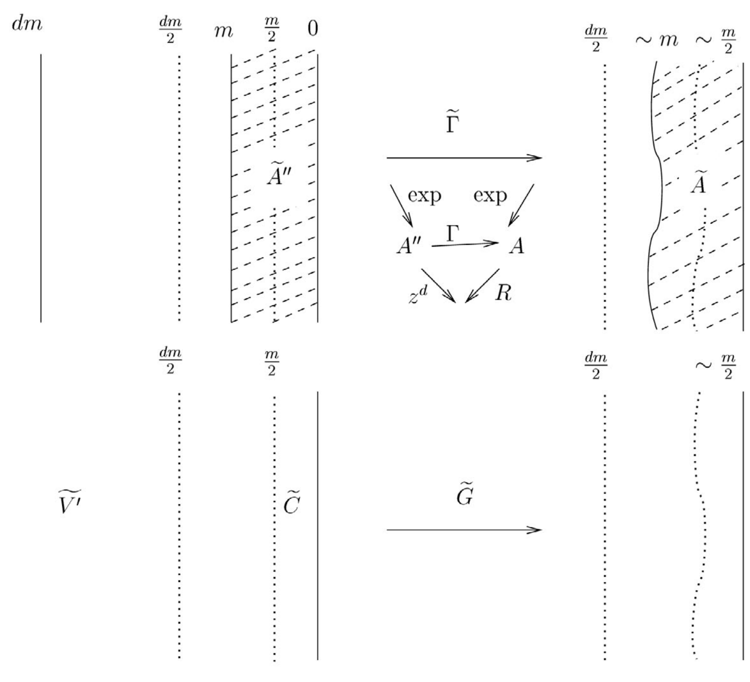
is the required map. To construct $\widetilde{G}$ we need only construct its values on the vertical strip $\{x+iy|x\in[\frac{dm}{2},\frac{m}{2}]\}$ so that it verifies (8) and such that $\widetilde{G}$ is continuous. The extension to this strip is the standard affine extension, i.e., it maps each horizontal segment affinely to the segment whose endpoints are determined by the values of $\widetilde{G}$ already defined. More precisely, set $\widetilde{\Gamma}(u)=u+\omega(u)$. For $x\in[a,b]:=[\frac{dm}{2},\frac{m}{2}]$ and $y\in\mathbb{R}$, we have

$$x+iy=\frac{b-x}{b-a}(a+iy)+\frac{x-a}{b-a}(b+iy).$$

Set

$$\widetilde{G}(x+iy)=\frac{b-x}{b-a}(a+iy)+\frac{x-a}{b-a}\widetilde{\Gamma}(b+iy)=x+iy+\frac{x-a}{b-a}\omega(b+iy).$$

The reader may check easily that $\widetilde{G}$ is a homeomorphism provided $|\omega'(b+iy)|<\frac{1}{4}$ for every $y\in\mathbb{R}$.



To give more precise estimates we write $\Gamma(z)=z(1+g(z))$ and $\Gamma^{-1}(z)= z(1+h(z))$. Then

$$\begin{aligned} z&=\Gamma^{-1}(\Gamma(z))=z(1+g(z))(1+h(\Gamma(z)),\\ |\log(1+g(z))|&=|\log(1+h(\Gamma(z))|. \end{aligned} \tag{10}$$

Assume now $a_1,\dots,a_d$ are the zeros of the Blaschke product R, we have

$$z^d(1+h(z))^d=R(z)=\prod_{j=1}^{d}\frac{1-\bar{a}_j}{1-a_j}\frac{z-a_j}{1-\bar{a}_j z},$$
$$\text{so}\quad (1+h(z))^d=\prod_{j=1}^{d}\frac{1-\bar{a}_j}{1-a_j}\frac{1-a_j/z}{1-\bar{a}_j z}.$$

and

$$|\log(1+h(z))|\le\frac{1}{d}\sum_{j=1}^{d}\left(\left|\log\left(1-\frac{a_j}{z}\right)\right|+|\log(1-\bar{a}_j z)|+|\log(1-\bar{a}_j)|+|\log(1-a_j)|\right).$$

As $|a_i|\le r$ and $|z|\ge\sqrt{r}$, and $|\log(1+v)|\le 2|v|$ for $|v|\le\frac{1}{2}$, there is $r_0>0$ so that if $r<r_0$,

$$|\log(1+h(z))|\le 2(\sqrt{r}+3r)\le 8\sqrt{r}.$$

Therefore

$$\forall u\in\widetilde{A}'',\quad |\omega(u)|=|\log(1+g(e^u))|\overset{(10)}{=}|\log(1+h(\Gamma(e^u)))|\le 8\sqrt{r}. \tag{11}$$

Now we use the Cauchy integral to estimate $\omega'(b+iy)$. Set $\rho=\frac{-m}{2}=-\frac{\log r}{2d}$. Note that $D(b+iy,\rho)\subset\widetilde{A}''$. So

$$|\omega'(b+iy)|\le\frac{8\sqrt{r}}{\rho}\longrightarrow_{r\to 0} 0. \tag{12}$$

We may thus adjust r_0 so that for $r<r_0$, $|\omega'(b+iy)|\le\frac{1}{4}$ and therefore $\widetilde{G}$ is a homeomorphism. We can readily estimate the Beltrami coefficient of $\widetilde{G}$: for $x\in]a,b[$,

$$\frac{\partial\widetilde{G}}{\partial x}(x+iy)=1+\frac{\omega(b+iy)}{b-a},\quad \frac{1}{i}\frac{\partial\widetilde{G}}{\partial y}(x+iy)=1+\frac{x-a}{b-a}\omega'(b+iy).$$

$$\frac{\partial\widetilde{G}}{\partial z}(x+iy)=\frac{1}{2}\left(\frac{\partial\widetilde{G}}{\partial x}(x+iy)+\frac{1}{i}\frac{\partial\widetilde{G}}{\partial y}(x+iy)\right)=1+\frac{\omega(b+iy)+(x-a)\omega'(b+iy)}{2(b-a)}$$

$$\frac{\partial\widetilde{G}}{\partial\bar{z}}(x+iy)=\frac{1}{2}\left(\frac{\partial\widetilde{G}}{\partial x}(x+iy)-\frac{1}{i}\frac{\partial\widetilde{G}}{\partial y}(x+iy)\right)=\frac{\omega(b+iy)-(x-a)\omega'(b+iy)}{2(b-a)}.$$

We may therefore adjust again r_0 so that if $r<r_0$, $|\frac{\partial \tilde{G}}{\partial z}(x+iy)|\geq 1/2$ for all $x\in]a,b[$. Note that $b-a=(d-1)\rho$. So

$$\left|\frac{\bar{\partial} F}{\partial F}\right| F \leq \sup_{x\in]a,b[}\left|\frac{\bar{\partial}\widetilde{G}}{\partial\widetilde{G}}\right| \leq 2\left|\frac{\partial\widetilde{G}}{\partial\bar{z}}\right| \leq \frac{|\omega(b+iy)|}{(d-1)\rho}+|\omega'(b+iy)|$$
$$\leq \frac{8\sqrt{r}}{\rho}+\frac{8\sqrt{r}}{\rho}=\frac{16d\sqrt{r}}{|\log(\sqrt{r})|}.$$

See [Sh] for a similar estimate. □

Proof of Theorem 4.2. Let us first show that the map $a\mapsto \widehat{v}(0,a)$ is injective on $\widetilde{H}^c_{e-1}$. It follows from Theorem 4.4 that if $\widehat{v}(0,a)=\widehat{v}(0,a')$, for some a, a', then $P_{e^{-1},a}$ and $P_{e^{-1},a'}$ are hybridly conjugate. The dynamics of $P_{e^{-1},a}$ and $P_{e^{-1},a'}$ are conformally conjugate on the immediate attracted basins of 0, by a unique biholomorphic map fixing the origin, because both basins are quadratic and the multipliers at the origin are identical. Hence essentially repeating the proof of Proposition 3.6 (see also the Remark following its proof) one proves that this hybrid equivalence coincides with a Möbius conjugacy on $\widetilde{B}_{e^{-1},a}(0)^c$. Details are left to the reader. It follows that $a=a'$.

To prove that each $\widehat{h}_{0,a}$ is injective, we prove that it has a quasi conformal extension to all of $\mathbb{C}$. As in the proof of Theorem 4.4 fix an arbitrary $a\in\widetilde{\mathcal{H}}^c_{e-1}$ and let H_λ, $\lambda\in\mathbb{D}$ be the hybrid conjugacies whose existence is assured by Theorem 4.4. Fix $\lambda\in\mathbb{D}$ and define $h=\widehat{h}_\lambda^{-1}\circ H_\lambda^{-1}\circ H_{e-1}$, so that each h is a hybrid equivalence from $P_{e^{-1}}$ to itself, which is tangent to the identity at ∞. Repeating once more the proof of Proposition 3.6 we find that h coincides with the identity on $\widetilde{B}_{e^{-1}}(0)^c$. Hence the two maps $\widehat{h}_\lambda$ and $H_\lambda^{-1}\circ H_{e-1}$ coincides on $\widetilde{B}_{e^{-1}}(0)^c$. Since H_λ^{-1} converges locally uniformly to the identity, it follows that $\widehat{h}_\lambda$ converges locally uniformly to the quasi conformal homeomorphism $H_e^{-1}:\mathbb{C}\longrightarrow\mathbb{C}$ on $\widetilde{B}_{e^{-1}}(0)^c$, from which the injectivity of $\widehat{h}_0=H_{e^{-1}}$ on $\widetilde{B}_{e^{-1}}(0)^c$ follows.

Finally,

$$\begin{aligned}
\widehat{H}(0,a,z)=\widehat{H}(0,a',z') &\Longrightarrow \widehat{v}(0,a)=\widehat{v}(0,a') \ \&\ \widehat{h}(0,a,z)=\widehat{h}(0,a',z')\\
&\Longrightarrow a=a' \ \&\ \widehat{h}(0,a,z)=\widehat{h}(0,a',z')\\
&\Longrightarrow a=a' \ \&\ \widehat{h}(0,a,z)=\widehat{h}(0,a,z')\\
&\Longrightarrow a=a' \ \&\ z=z'.
\end{aligned}$$

So $\widehat{H}$ is injective. □

4.2. Lavaurs motion. The following example is different from the first and most other applications of the BH-motion. It is a motion of a two generator dynamical system (P, g_σ) consisting of a center attracting dynamics $(P, \bar{\mathbb{C}}, \infty)$ with $P(z)=z^2+\frac{1}{4}$ (which is invariant under the BH-motion, by Proposition 3.6) and a so called parabolic enrichment or Lavaurs map g_σ coming from the complementary parabolic basin $B(0)$. Let $\phi : B(\infty) \longrightarrow \bar{\mathbb{C}} \backslash \overline{\mathbb{D}}$ denote the Böttcher coordinate at ∞ and let ψ denote its inverse. Let $\Phi : B(0) \longrightarrow \mathbb{C}$ denote an attracting Fatou coordinate and $\Psi : \mathbb{C} \longrightarrow \mathbb{C}$ denote a repelling Fatou parameter, i.e., $\Phi \circ P(z) = 1 + \Phi(z)$ and $P \circ \Psi(z) = \Psi(z+1)$ where defined. We shall normalize Φ and Ψ by $\Phi(0) = 0$ and $\Psi(0) = \phi^{-1}(e)$. Define the Lavaurs map $g_\sigma : B(0) \longrightarrow \mathbb{C}$ of phase $\sigma \in \mathbb{C}$ by $g_\sigma(z) = \Psi \circ T_\sigma \circ \Phi$, where $T_\sigma(z) = z + \sigma$ and let $\Sigma = \{\sigma \in \mathbb{C} \mid g_\sigma(0) \in B(\infty)\} = \Psi^{-1}(B(\infty)) \ni 0$. The Julia set $J(P, g_\sigma)$ of the Lavaurs enriched dynamical system is the closure of $\cup_{n,m \geq 0} P^{-n}(g_\sigma^{-m}(J_P))$.

Theorem 4.6. *There exists a holomorphic motion $h : \Sigma \times \bar{\mathbb{C}} \longrightarrow \bar{\mathbb{C}}$, such that $h_\sigma \circ P = P \circ h_\sigma$ and $h_\sigma \circ g_0 = g_\sigma \circ h_\sigma$.*

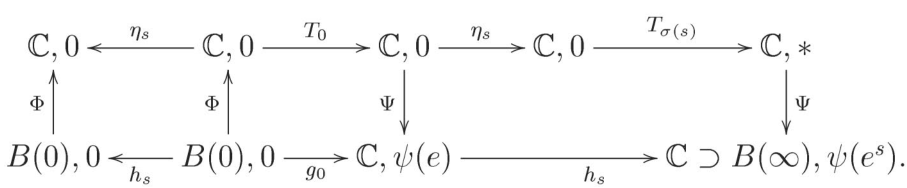
The map h_σ fixes J_P point-wise, but moves the points of the enrichment. However the the enriched Julia set is 1 periodic as a compact set, because $g_{\sigma+1} = P \circ g_\sigma = g_\sigma \circ P$ and hence the two enriched dynamical systems (P, g_σ) and $(P, g_{\sigma+1})$ have the same enriched Julia set.

Proof. Denote by $\psi : \bar{\mathbb{C}} \backslash \overline{\mathbb{D}} \longrightarrow B(\infty)$ the inverse of ϕ, the Böttcher parameter at ∞. We consider a BH-motion of the enriched dynamical system (P, g_0) based at the super attracting fixed point ∞. More precisely let μ_s denote the unique Beltrami form which equals $(l_s \circ \phi)^*(\mu_0)$ on $B(\infty)$, where $\mu_0 = 0$ is the zero or standard Beltrami form and which is invariant under the enriched dynamical system (P, g_0), i.e., $P^*(\mu_s) = \mu_s$ and $g_0^*(\mu_s) = \mu_s$. Note that μ_s is also supported in $B(1/2)$, because g_0 maps part of $B(1/2)$ into $B(\infty)$.

Let $h_s : \bar{\mathbb{C}} \longrightarrow \bar{\mathbb{C}}$ denote the solution of the Beltrami equation $\bar{\partial} h = \mu_s \partial h$ normalized by $h_s(0) = 0$, $h_s(1/2) = 1/2$ and $h_s(\infty) = \infty$. Then $f = h_s \circ P \circ h_s^{-1}$ is a centered quadratic polynomial with a parabolic fixed point of multiplier 1 at $1/2$. There is only one such polynomial, it is P. So h_s conjugates P to itself.

Define $\widehat{\mu}_s = \Psi^*(\mu_s)$ on $\mathbb{C}$, then $T_1^*(\widehat{\mu}_s) = \widehat{\mu}_s$. Let $\eta : \bar{\mathbb{C}} \longrightarrow \bar{\mathbb{C}}$ denote the solution of the Beltrami equation $\bar{\partial} f = \widehat{\mu}_s \partial f$ normalized by $\eta_s(0) = 0$, $\eta_s(1) = 1$ and $\eta_s(\infty) = \infty$. We have $\eta_s(z+1) = \eta_s(z) + 1$, because $\widehat{\mu}_s$ is 1-periodic.

For each s the map $\eta_s \circ \Phi \circ h_s^{-1}$ is holomorphic, conjugates P to translation by 1 and fixes 0. Hence it equals Φ by uniqueness of normalized Fatou coordinates. Arguing similarly we find that $h_s \circ \Psi \circ \eta_s^{-1} = \Psi \circ T_{\sigma(s)}$ for some complex number $\sigma(s) \in \Sigma$. We have $h_s \circ \Psi \circ \eta_s^{-1}(0) = h_s(\psi(e)) = \psi(e^s)$ by the normalization of Ψ. Thus $\sigma(s)$ can be



chosen to depend continuously on s. Furthermore $h_s \circ g_0 = g_\sigma \circ h_s$. See the following diagram.

The restriction $\Psi : \Sigma \longrightarrow \mathrm{B}(\infty)$ is a universal covering, (see e.g. [P]) with $\Psi(0) = \psi(\mathrm{e})$. And $\psi \circ \exp : \mathbb{H} \longrightarrow \mathbb{B}(\infty)$ is also a universal covering, but with $\psi \circ \exp(1) = \psi(e)$. Hence there exists a unique lift $\widehat{\sigma} : \mathbb{H} \longrightarrow \Sigma$ of $\psi \circ \exp$ to Ψ with $\widehat{\sigma}(1) = 0$. This lift is an isomorphism, since both coverings are universal, and it satisfies $\widehat{\sigma}(2s) = \widehat{\sigma}(s) + 1$. However since $\Psi \circ T_{\widehat{\sigma(s)}} = \Psi \circ T_{\sigma(s)}$ and both functions σ and $\widehat{\sigma}$ are continuous, we have $\sigma = \widehat{\sigma}$. As σ is an isomorphism from $\mathbb{H}$ to Σ we may use σ as a parameter for the holomorphic motion, replacing the parameter space $\mathbb{H}$ by Σ and the base point 1 by 0. □

Acknowledgements. We would like to thank Institut Henri Poincaré, Paris and the organizers of its trimester program on holomorphic dynamics during the autumn 2003 for hospitality and stimulating activities. The first author would also like to thank CNRS-UMR 8088 at Université de Cergy-Pontoise for its hospitality during the creation of this paper.

References

[BH1] B. Branner & J.H. Hubbard, The iteration of cubic polynomials. Part I: The global topology of parameter space, Acta Mathematica 160 (1988), 143–206.

[D] A. Douady. Conjectures about the Branner-Hubbard motion of Cantor sets in $\mathbb{C}$ in this Volume.

[DH1] A. Douady & J.H. Hubbard, Étude dynamique des polynômes complexes, I, publications mathématiques d'Orsay, 84–02, 1984.

[DH2] A. Douady & J.H. Hubbard, On the dynamics of polynomial-like mappings, Ann. Scient. Ec. Norm. Sup., t.18, p. 287–343, 1985.

[KN] Komori, Y. & Nakane, Sh., Landing property of stretching rays for real cubic polynomials, Conformal geom. and dyn. 8 (2004), 87–114.

[Mi] J. Milnor, Dynamics in One Complex Variable, Introductory Lectures, vieweg, 1999.

[P] C. L. Petersen, *On the Pommerenke-Levin-Yoccoz inequality*, Ergod. Th. & Dynam. Sys., 13 (1993) 785–806.

[P-T] C. L. Petersen and Tan Lei, *BH-deformations of attracting dynamics*, manuscript in preparation.

[Sh] M. Shishikura, *Bifurcation of parabolic fixed points*, in "The Mandelbrot set, Theme and Variations", Ed. Tan Lei, LMS Lect. Note Ser. 274, Cambridge Univ. Press, 2000, p. 325–363.

[Ta] Tan Lei, Stretching rays and their accumulations, following Pia Willumsen, in this volume.

[Ri] S. Rickman, Removability theorems for quasi-conformal mappings, Ann. Ac. Scient. Fenn. 449, p. 1–8, 1969.

[U] E. Uhre, Construction of a Holomorphic Motion in Part of the Parameter Space for a Family of Quadratic Rational Maps. Master Thesis, IMFUFA, RUC, Denmark 2004.

[Wi] P. Willumsen, Holomorphic dynamics, On accumulation of stretching rays, Ph.D. thesis, Danmarks Tekniske Universitet, 1997.

IMFUFA
Roskilde University
Universitetsvej 1
DK-4000 Roskilde
e-mail: lunde@ruc.dk

Unité CNRS-UMR 8088,
Département de Mathématiques,
Université de Cergy-Pontoise,
2 Av. A. Chauvin, 95302 Cergy-Pontoise.
e-mail: tanlei@math.u-cergy.fr

Examples of Feigenbaum Julia sets with small Hausdorff dimension

Artur Avila and Mikhail Lyubich

To Bodil Branner on her 60th birthday

Abstract. We give examples of infinitely renormalizable quadratic polynomials $F_c : z \mapsto z^2 + c$ with stationary combinatorics whose Julia sets have Hausdorff dimension arbitrary close to 1. The combinatorics of the renormalization involved is close to the Chebyshev one. The argument is based upon a new tool, a "Recursive Quadratic Estimate" for the Poincaré series of an infinitely renormalizable map.

1. Introduction. One of the most remarkable objects in complex dynamics are the fixed points of the Douady-Hubbard renormalization operator. Such objects have a distinguished place in the dictionary between rational maps and Kleinian groups (see [Mc2]). Existence of the renormalization fixed points established in the works of Sullivan [S2] and McMullen [Mc2] (under certain assumptions) implies many beautiful features (self-similarity, universality, hairyness, ...) of Feigenbaum Julia sets (see §2.2 for the definition). However, even with this thorough information, some basic questions concerning measure and dimension of these Julia sets have remained unsettled.

One of the key questions (asked, for instance, in [Mc2]) regarding the geometry of Feigenbaum Julia sets has been the following: *Is the Hausdorff dimension of a Feigenbaum Julia set always equal to* 2? In [AL] we supply a fairly large class of Feigenbaum Julia sets with $\mathrm{HD}(J) < 2$, thus giving a negative answer to the above question (though it is still unknown whether there exist Feigenbaum Julia sets with Hausdorff dimension 2). In this paper we show that in fact the dimension of a Feigenbaum Julia set can be arbitrarily close to 1:

Theorem 1.1. *There exists a sequence of Feigenbaum quadratic polynomial* $F_p \equiv F_{c_p}$: $z \mapsto z^2 + c_p$ *with* $c_p \in \mathbb{R}$, $c_p \to -2$, *such that* $\mathrm{HD}(J(F_p)) \to 1$ *as* $p \to \infty$.

Hausdorff dimension is closely related to another geometric characteristic of the Julia set, the *critical exponent* δ_{cr} (see §2.3). In fact, for a Feigenbaum map F_c,

$$\mathrm{HD}(J(F_c)) = \delta_{\mathrm{cr}}(J(F_c)),$$

Date: August 5, 2005.

provided area $(J(F_c)) = 0$ [AL], and the same is true for the associated renormalization fixed point f_c. In fact, it follows from Bishop's work [B] that for any c, $\mathrm{HD}(J(F_c)) \le \delta_{\mathrm{cr}}(J(F_c))$, provided area $(J(F_c)) = 0$. This allows us to reduce Theorem 1.1 to the following two results.

Theorem 1.2. *Let $f_p \equiv f_{c_p}$ be the fixed point of the renormalization operator of period p with combinatorics closest to the Chebyshev one. Then $\delta_{\mathrm{cr}}(f_p) \to 1$ as $p \to \infty$.*

The proof of this theorem is based upon a "Recursive Quadratic Estimate" for the Poincaré series which provides a new efficient tool for getting bounds on the critical exponent.

Theorem 1.3. *For large p,* $\mathrm{area} J(f_p) = 0$.

Remark. (1) The class of Feigenbaum maps with $\mathrm{HD}(J(f)) < 2$ supplied in [AL] is qualitatively the same as the class treated in Yarrington's thesis [Y] (see also §9 of [AL]) for which $\mathrm{area}(J) = 0$ (which in turn, is qualitatively the same as the class of infinitely renormalizable maps for which a *priori* bounds were established in [L2]). Though Theorem 1.3 is not formally covered by [AL, Y], it is proved by a similar method, which becomes more direct in our situation. Similarly, to prove Theorem 1.2 we adjust the method of [AL] to the Chebyshev combinatorics, which makes it (in this combinatorial case) simpler and more powerful.

(2) In Theorem 1.2, we restrict ourselves to a very particular sequence of combinatorics converging to the Chebyshev one, though the result should still hold for somewhat more general combinatorics. Obviously, *some* restrictions on the combinatorics are necessary: There are renormalization fixed points f_p, with combinatorics arbitrarily close to the Chebyshev one, and such that $\mathrm{HD}(J(f))$ is not close to 1. This holds even for real combinatorics, and can be seen, e.g., by considering parabolic bifurcations.

2. Basic concepts.

2.1. Notations. $\mathbb{D}_r(z) \equiv \{w \in \mathbb{C},\ |w - z| < r\}$, $\mathbb{D}_r \equiv \mathbb{D}_r(0)$. A domain is a connected open subset of $\mathbb{C}$. A topological disk is a simply connected domain. $U \Subset V$ means that U is *compactly contained* in V.

Notation $a \asymp b$ means that $C^{-1} < a/b < C$ with a constant $C > 0$ independent of particular a and b under consideration; $a \approx b$ means that a is close to b.

We usually denote the p-fold iterate of a map f by f^p, but occasionally use a more forceful notation $f^{\circ p}$.

Let $\omega(x) \equiv \omega_f(x) = \cap_{m \ge 0} \overline{\{f^k(x),\ k \ge m\}}$ denote the *ω-limit set* of x.

For a quadratic-like map $f : U \to V$ (see below) with the critical point at 0, let $\mathcal{O}(f) \equiv \overline{\{f^k(0),\ k > 0\}}$ denote its *postcritical set*.

2.2. Quadratic-like maps and renormalization. A *quadratic-like map* is a holomorphic double covering map $f : U \to V$ where $U, V \subset \mathbb{C}$ are topological disks and $U \Subset V$. Such a map has a unique critical point which we will assume to be 0. Let

$K(f) \equiv \cap_{k=0}^{\infty} f^{-k}(U)$ denote the *filled Julia set* of f and let $J(f) \equiv \partial K(f)$ denote its *Julia set*.

Two quadratic-like germs f and g are said to be *hybrid equivalent* if there exists a quasiconformal map $h : \mathbb{C} \to \mathbb{C}$ satisfying $h(f(x)) = g(h(x))$ for x near $J(f)$ such that $\bar{\partial}h|J(f) = 0$. Any quadratic-like map $f : U \to V$ with connected Julia set is hybrid equivalent to a unique quadratic polynomial $F : z \mapsto z^2 + c$ called the *straightening* of f [DH]. Moreover, the dilatation $\mathrm{Dil}(h)$ of the (appropriately chosen) conjugacy h depends only on $\mathrm{mod}(V \backslash U)$, and $\mathrm{Dil}(h) \to 1$ as $\mathrm{mod}(V \backslash U) \to \infty$.

The Julia set $J(f)$ of a quadratic-like map is either connected or Cantor. If $J(f)$ is connected, there exists a unique repelling or parabolic fixed point $\beta = \beta(f) \in J(f)$ such that $J(f) \backslash \{\beta(f)\}$ is connected. The other fixed point is denoted by $\alpha = \alpha(f)$. We will only consider quadratic-like maps with connected Julia set.

A quadratic-like map which is considered only up to choice of domains is called a *quadratic-like germ*. More precisely, one says that two quadratic-like maps with connected Julia sets represent the same germ if they have a common Julia set and coincide in a neighborhood of it. We shall consider quadratic-like germs up to affine conjugacy. For a germ f, we let $\mathrm{mod}(f) = \sup \mathrm{mod}(V \backslash U)$, where the suppremum is taken over all possible choices of domains U and V.

A quadratic-like map $f : U \to V$ is called *renormalizable* with period $p > 1$ if there exist topological disks $U' \Subset V'$ containing the critical point such that

1. $g \equiv f^p : U' \to V'$ is a quadratic-like map with connected Julia set $J(f')$ called a *pre-renormalization* of f;
2. For every $1 \leq k \leq p-1$, either $f^k(J(g)) \cap J(g) = \emptyset$ or $f^k(J(g)) \cap J(g) = \{\beta(g)\}$.

The renormalization operator R is defined on the space of germs by letting $Rf = g$. The minimal $p = p(f) > 1$ for which f is renormalizable is called the *renormalization period* of f. In what follows, the operator R will always correspond the this minimal period.

We will be concerned with *infinitely renormalizable* quadratic-like maps f (so that all the renormalizations $R^m f$, $m \geq 0$, are well defined). If the periods of all the renormalizations $R^m f$ are bounded, we say that f has a *bounded type*. If $\mathrm{mod}(R^m f) \geq \epsilon > 0$, $m \geq 0$, we say that f has *a priori* bounds. An infinitely *renormalizable map of bounded type with a priori* bounds will be also called a *Feigenbaum map*. The notion of a Feigenbaum map is invariant under hybrid equivalence.

A quadratic-like map $f : U \to V$ is said to be a *renormalization fixed point* if f is renormalizable and $Rf = f$. In other words, $f^p(x) = \lambda f(\lambda^{-1}x)$ near $J(g)$ for some $\lambda \in \mathbb{D} \backslash \{0\}$, where p is the renormalization period of f and g is a pre-renormalization of f. Such maps are automatically Feigenbaum.

2.3. Poincaré series. Let $f : U \to V$ be a quadratic-like map.

Sullivan's Poincaré series [S1] is defined as follows:

$$\Xi_\delta(z) = \sum_{k=0}^{\infty} \sum_{f^k(w)=z} |Df^k(w)|^{-\delta}, \quad z \in V \backslash \mathcal{O}(f), \quad \delta > 0.$$

It follows from the Koebe Distortion Theorem that $\Xi_\delta(z) \le C(z, z')^\delta\, \Xi_\delta(z')$ for any $z, z' \in V \setminus \mathcal{O}(f)$. In particular, Ξ_δ is finite or infinite independently of z.

The function $\delta \mapsto \Xi_\delta$ is obviously convex. By definition, the *critical exponent*, $\delta_{\text{cr}}(f) \in [0, \infty]$, is the unique value of δ that separates convergent Ξ_δ from divergent ones. The critical exponent depends only on the germ of f near $K(f)$.

It is easy to see that Ξ_2 is always finite (area argument) and, since $J(f)$ is assumed to be connected, $\Xi_1 = \infty$ (length argument), see §2.9 of [AL]. Thus we actually have $\delta_{\text{cr}}(f) \in [1, 2]$. In fact, $\delta_{cr} > 1$, unless $J(f)$ is a real analytic curve.

2.3.1. Poincaré series for subfamilies of orbits. An *orbit of length* $k \ge 0$ is a sequence $(x_0, \dots, x_k)$, where $x_k \in V$ and $f(x_i) = x_{i+1}$ for $0 \le i < k$. An orbit of zero length is called *trivial*.

Given a family $\mathcal{F}$ of orbits $(x_0, \dots, x_k)$, we define a function $\mathbb{C} \to [0, \infty]$

$$\Xi_\delta(\mathcal{F})(z) = \sum_{k=0}^{\infty} \sum_{(x_0, \dots, x_k = z) \in \mathcal{F}} |Df^k(x_0)|^{-\delta}$$

(to keep notation shorter, we do not explicitly mention f). Let $\Xi_\delta^{[j]}$ denote the truncation of Ξ_δ at level j,

$$\Xi_\delta^{[j]}(\mathcal{F})(z) = \sum_{k=0}^{j} \sum_{(x_0, \dots, x_k = z) \in \mathcal{F}} |Df^k(x_0)|^{-\delta},$$

with convention that $\Xi^{[j]} = 0$ for $j < 0$. Note that $\Xi^{[0]}(\mathcal{F})$ is equal to 1 or 0 depending on whether $\mathcal{F}$ contains the trivial orbit or not.

2.3.2. Arrow notation. Let us introduce a convenient notation for certain families of orbits. Let $D, E \subset V$, $S \subset U$. By $D \leftarrow E$, we will understand the family of orbits $(x_0, \dots, x_k)$ with $x_0 \in E$ and $x_k \in D$. The family of orbits $(x_0, \dots, x_k)$ with $x_0 \in E$, $x_k \in D$ and $x_1, \dots, x_{k-1} \in S$ will be denoted $D \underset{S}{\leftarrow} E$. A "plus sign" over the arrow will indicate that only non-trivial orbits are considered. The juxtaposition of arrows will denote composition in the natural way. For instance,

$$D \overset{+}{\underset{S}{\leftarrow}} D \underset{S}{\leftarrow} E,$$

denotes the family of orbits $(x_0, \dots, x_k)$, with $x_0 \in E$, $x_k \in D$, such that $x_i \in D$ for some $0 \le i < k$, and $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{k-1} \in S$.

3. Quadratic recursive estimate. We will now introduce a version of the Quadratic Recursive Estimate which is sufficient for purposes of this paper (see [AL] for a finer version). We shall restrict ourselves to the case of renormalization fixed points. The argument is based on a combinatorial breakdown of orbits which exploits the scaling self-similarity of the dynamics.

Let $f : U \to V$ be a fixed point of renormalization of period p, $f^p(x) = \lambda f(\lambda^{-1}x)$ near 0. Let $U' = \lambda U$, $V' = \lambda V$, and let $g \equiv f^p : U' \to V'$. Let $A = V \setminus U$, $A' = V' \setminus U'$. We assume that $V' \subset U$, g is the first return from U' to V', and that $\mathcal{O}(f)$ does not intersect $\bar{A}'$.

Lemma 3.1. *Let* $s_j(\delta) = \sup_{z \in A'} \Xi_\delta^{[j]}(A' \leftarrow U)(z)$. *Then*

$$s_{j+1}(\delta) \le P_\delta(s_j(\delta)),$$

where $x \mapsto P_\delta(x)$ *is a quadratic polynomial with positive coefficients which can be expressed explicitly in terms of the Poincaré series* $\Xi_\delta(\mathcal{F})$ *over families* $\mathcal{F}$ *of orbits that do not accumulate on 0.*

If P_δ *has a positive fixed point* s *then*

$$\sup_{z \in A'} \Xi_\delta(A' \leftarrow U)(z) = \lim s_j \le s,$$

so that $\delta_{\mathrm{cr}}(f) \le \delta$.

Proof. In what follows, the sup is always taken over z, the terminal point of the orbit in question. We will also omit the truncation parameter (j or $j+1$) in the notation.

We can decompose $A \leftarrow U$ into two groups: $A \underset{U \setminus V'}{\longleftarrow} U \setminus V'$ and $A \underset{U \setminus V'}{\longleftarrow} A' \leftarrow U$. This gives the inequality

$$\begin{aligned} \sup \Xi_\delta(A \leftarrow U) \le{}& \sup \Xi_\delta(A \underset{U \setminus V'}{\longleftarrow} U \setminus V') \\ &+ \sup \Xi_\delta(A \underset{U \setminus V'}{\longleftarrow} A') \sup \Xi_\delta(A' \leftarrow U). \end{aligned} \tag{3.1}$$

In turn, we can decompose $A' \overset{+}{\leftarrow} U$ into two groups:

1. $A' \underset{U \setminus A'}{\longleftarrow} U \setminus A'$, which can be further decomposed into

$$A' \underset{U \setminus V'}{\longleftarrow} U \setminus V', \quad A' \underset{U \setminus A'}{\longleftarrow} U', \quad \text{and} \quad A' \underset{U \setminus A'}{\longleftarrow} U' \underset{U \setminus V'}{\longleftarrow} U' \setminus V';$$

2. $A' \underset{U \setminus A'}{\overset{+}{\longleftarrow}} A' \leftarrow U$, which can be further decomposed into

$$A' \underset{U \setminus V'}{\overset{+}{\longleftarrow}} A' \leftarrow U \quad \text{and} \quad A' \underset{U \setminus A'}{\longleftarrow} U' \underset{U \setminus V'}{\longleftarrow} A' \leftarrow U.$$

This gives the following inequality

$$\begin{aligned}\sup \Xi_\delta(A' \leftarrow U) \leq 1 &+ \sup \Xi_\delta(A' \underset{U\setminus V'}{\longleftarrow} U\setminus V') \\ &+ \sup \Xi_\delta(A' \underset{U\setminus A'}{\longleftarrow} U')(1 + \sup \Xi_\delta(U' \underset{U\setminus V'}{\longleftarrow} U\setminus V')) \\ &+ \sup \Xi_\delta(A' \overset{+}{\underset{U\setminus V'}{\longleftarrow}} A') \sup \Xi_\delta(A' \leftarrow U) \\ &+ \sup \Xi_\delta(A' \underset{U\setminus A'}{\longleftarrow} U') \sup \Xi_\delta(U' \underset{U\setminus V'}{\longleftarrow} A') \\ &\quad \sup \Xi_\delta(A' \leftarrow U), \end{aligned} \tag{3.2}$$

where the first term, 1, accounts for the trivial orbits.

Notice that since $x \mapsto f^p x$ is the first return map from U' to V', if $(x_0, \dots, x_k)$ belongs to $A \leftarrow U$ then $(\lambda x_0, \dots, f^{kp}(\lambda x_0))$ belongs to $A' \underset{U\setminus A'}{\longleftarrow} U'$. This correspondence is readily seen to be a bijection between $A \leftarrow U$ and $A' \underset{U\setminus A'}{\longleftarrow} U'$ preserving the weights of the Poincaré series. Hence

$$\Xi_\delta(A \leftarrow U)(x) = \Xi_\delta(A' \underset{U\setminus A'}{\longleftarrow} U')(\lambda x)$$

and

$$\sup \Xi_\delta(A' \underset{U\setminus A'}{\longleftarrow} U') = \sup \Xi_\delta(A \leftarrow U). \tag{3.3}$$

Plugging (3.1) into (3.3), and then plugging the resulting expression for $\sup \Xi_\delta(A' \underset{U\setminus A'}{\longleftarrow} U')$ into the 2nd and 4th lines of (3.2), we obtain

$$\sup \Xi_\delta(A' \leftarrow U) \leq \alpha + \beta \sup \Xi_\delta(A' \leftarrow U) + \gamma \sup \Xi_\delta(A' \leftarrow U)^2,$$

where

$$\begin{aligned}\alpha = 1 &+ \sup \Xi_\delta(A' \underset{U\setminus V'}{\longleftarrow} U\setminus V') + \sup \Xi_\delta(A \underset{U\setminus V'}{\longleftarrow} U\setminus V') \\ &(1 + \sup \Xi_\delta(U' \underset{U\setminus V'}{\longleftarrow} U\setminus V')),\end{aligned} \tag{3.4}$$

$$\begin{aligned}\beta = \sup \Xi_\delta(A' \overset{+}{\underset{U\setminus V'}{\longleftarrow}} A') &+ \sup \Xi_\delta(A \underset{U\setminus V'}{\longleftarrow} A')(1 + \sup \Xi_\delta(U' \underset{U\setminus V'}{\longleftarrow} U\setminus V')) \\ &+ \sup \Xi_\delta(A \underset{U\setminus V'}{\longleftarrow} U\setminus V') \sup \Xi_\delta(U' \underset{U\setminus V'}{\longleftarrow} A'),\end{aligned} \tag{3.5}$$

and

$$\gamma = \sup \Xi_\delta(A \underset{U\setminus V'}{\longleftarrow} A') \sup \Xi_\delta(U' \underset{U\setminus V'}{\longleftarrow} A'). \tag{3.6}$$

This is the desired quadratic recurrence estimate for $\sup \Xi_\delta(A' \leftarrow U)$. The above three formulas give an explicit expression of the coefficients α, β and γ of P_δ in terms of Poincaré series over families of orbits that do not accumulate on 0.

For the last statement, notice that $s_j \leq P^j(s_{-1}) = P^j(0) \leq s$ for all j. Thus, for every $z \in A'$ we have $\Xi_\delta(z) \leq \sup \Xi_\delta(A' \leftarrow U) = \lim_{j\to\infty} s_j \leq s$, which shows that $\delta_{\text{cr}}(f) \leq \delta$. $\square$

4. Renormalization with combinatorics closest to Chebyshev. In this section we will show that the critical exponent of maps with combinatorics "close to Chebyshev" can be arbitrarily close to 1. Our bounds on the critical exponent will be based on direct estimates of the coefficients of the quadratic recursive polynomial corresponding to a nearly Chebyshev map.

4.1. Basic properties. Let $\text{ч}(x) = 2 - x^2$ be the Chebyshev polynomial. Let f_p *be the fixed point of the renormalization operator of period p, with (real) combinatorics closest to Chebyshev*: f_p is combinatorially characterized among fixed points of renormalization of period p by being (up to affine conjugacy) a real-symmetric quadratic-like germ such that $f_p(0) > 0$ and $f_p^i(0) < 0$, $1 < i < p$. The existence of f_p is a particular case of a result of Sullivan [MS].

We normalize f_p so that its orientation preserving fixed point is -2. Let $-1 < \lambda_p < 0$ be the scaling factor of f_p. Then we have near zero:

$$g_p := f_p^{\circ p}(x) = \lambda_p f_p(\lambda_p^{-1} x).$$

Notice that $[-2, 2] \subset J(f_p)$ and $f_p : [-2, 2] \to [-2, 2]$ is a unimodal map. Let $\alpha_p > 0$ stand for he orientation reversing fixed point of f_p.

A basic fact is that all of the f_p belong to some fixed *Epstein class*, that is, there exists $\epsilon > 0$ such that $f_p : [-2, 2] \to [-2, 2]$ extends to a real-symmetric double covering onto the slit plane $\mathbb{C}\backslash(\mathbb{R}\backslash(-2-\epsilon, 2+\epsilon))$. (The natural topology in such an Epstein class makes it a compact space.) This is a consequence of the real *a priori* bounds, see [MS]. This yields a number of nice properties of the maps f_p. The ones that are relevant for us are summarized in the following lemma:

Lemma 4.1. *Let* $p \geq 3$, $T' = (-\alpha_p, \alpha_p)$, $\mathbb{V}' = \{z : |z| < \alpha_p\}$, *and let* $\mathbb{U}'$ *be the component of* $f_p^{-p}(\mathbb{V}')$ *containing* 0. *Let* $\mathbb{U} = \lambda^{-1}\mathbb{U}'$ *and* $\mathbb{V} = \lambda^{-1}\mathbb{V}'$. *Then*

(1) f_p *extends to a double covering onto the slit plane* $\mathbb{C}\backslash(\mathbb{R}\backslash T)$*;*
(2) $f_p \to \text{ч}$ *uniformly in* $[-2, 2]$ *(in particular* $f_p(0) \to 2$ *and* $\alpha_p \to 1$*);*
(3) *the maps* $g_p : \mathbb{U}' \to \mathbb{V}'$ *and* $f_p : \mathbb{U} \to \mathbb{V}$ *are quadratic-like for p sufficiently large;*
(4) $\text{mod}(\mathbb{V}\backslash\overline{\mathbb{U}}) = \text{mod}(\mathbb{V}'\backslash\overline{\mathbb{U}'}) \to \infty$*;*
(5) $\lambda_p \to 0$.

Proof. Let $S_k \subset [-2, 2]$ be the component of $(f_p|[-2, 2])^{-(p-k)}(T')$ containing $f_p^k(0)$, $k = 0, 1, \ldots, p$. Since the intervals $[-2, -\alpha_p]$ and $[\alpha_p, 2]$ are monotonically mapped by f_p onto $[-2, \alpha_p]$, the maps $f_p : S_k \to S_{k+1}$ are diffeomorphisms for $k = 1, 2, \ldots, p-1$. This implies the first assertion by rescaling.

The second assertion follows from the compactness of the Epstein class and the first assertion.

Moreover, $|S_1|^{-1/p} \approx \operatorname{dist}(S_1, f_p(0))^{-1/p} \approx 4$, where "4" is the multiplier of the orientation preserving fixed point -2 of $\mathfrak{q}$. Since f_p belongs to the Epstein class, the component of $f_p^{-(p-1)}(\mathbb{V}')$ containing $f_p(0)$ is contained in the round disk with diameter S_1. Hence $(\operatorname{diam}\ \mathbb{U}')^{-1/p} \approx 2$, which implies assertions (3) and (4) for g_p. The corresponding assertions for f_p are obtained by rescaling.

Since

$$\lambda_p = \frac{\operatorname{diam}\ J(g_p)}{\operatorname{diam}\ J(f_p)} \leq \frac{1}{4}\ \operatorname{diam} \mathbb{U}',$$

assertion (5) follows, too. □

4.2. Estimates for the coefficients. We will now use the information provided by Lemma 4.1 to give direct estimates on the coefficients of the quadratic recursive estimate.

The following lemma gives control of expansion along the orbits that stay away from 0:

Lemma 4.2. *For every $x \in \mathbb{C}\setminus\{-2, 2\}$, there exists $K = K(x)$ with the following properties:*

(1) *If $\mathfrak{q}^m(y) = x$, $m \geq 1$, then $|D\,\mathfrak{q}^m(y)| \geq K2^m$;*
(2) *For any $\epsilon > 0$ and $p \geq p_0(\epsilon)$, if $x \in \frac{1}{2}\mathbb{V}$ and $f_p^m(y) = x$, $m \geq 1$, with $f_p^k(y) \notin \mathbb{D}_\epsilon$, $0 \leq k \leq m-1$, then $|Df_p^m(y)| \geq K(2-\epsilon)^m$.*

Moreover, K depends only on the distance from x to $\{-2, 2\}$ and goes to infinity as x goes to infinity.

Proof. Consider the map $T : \mathbb{C}\setminus\mathbb{D} \to \mathbb{C}$, $T(z) = -(z + z^{-1})$ semi-conjugating $z \mapsto z^2$ to $\mathfrak{q}$; $T(z^2) = \mathfrak{q}(T(z))$. If $x = T(x')$, $y = T(y')$ and $\mathfrak{q}^m(y) = x$ with $m \geq 1$, then $D\mathfrak{q}^m(y) = DT(x')DT(y')^{-1}2^m x' y'^{-1}$. Since $|y'| = |x'|^{1/2^m} \leq \sqrt{|x'|}$, we have:

$$|D\mathfrak{q}^m(y)| \geq \frac{|DT(x')|}{|DT(y')|}|x'|^{1/2}2^m. \tag{4.1}$$

Since $|DT(y')| \leq 2$ for all $y' \in \mathbb{C}\setminus\mathbb{D}$ and $|DT(x')|$ is bounded away from zero for x outside a neighborhood of $\{-2, 2\}$, (4.1) implies (1).

Since the dynamics of $\mathfrak{q}$ outside a neighborhood of 0 is hyperbolic and hence Hölder stable, the second statement follows easily from Lemma 4.1. □

For $0 < \rho < 1$, let $V' = V'_\rho = \mathbb{D}_\rho$ and $U' = U'_{\rho,p} = f_p^{-p}(V')|0$. It follows from Lemma 4.1 that for $p > p_0(\rho)$, the map $g_p = f_p^{\circ p} : U' \to V'$ is a quadratic-like prerenormalization of $f_p : U \to V$, where $U = U_{\rho,p} = \lambda_p^{-1}U'$ and $V = V_{\rho,p} = \lambda_p^{-1}V'$. *In what follows, ρ and p will be usually suppressed in the notation.*

The following two lemmas give control of expansion along the orbits that originate near 0.

Lemma 4.3. *For every* $0 < \rho \le 1/10$, $0 < \kappa \le 1/10$, *and* $p > p_0(\kappa, \rho)$, *we have*

$$|f(y) - 2| \le |y|^{2-\kappa} \tag{4.2}$$

for any $y \in \mathbb{D}_{e^{-\kappa^{-2}}} \setminus U'$.

Proof. Since the map $f^{p-1} : [f(0), 2] \to [f^p(0), -2]$ has bounded distortion,

$$|f(0) - 2| \asymp |Df^{p-1}(f(0))|^{-1}.$$

Let $W = W(p, \rho)$ be the connected component of $f^{-(p-1)}(V')$ containing $f(0)$. Similarly, since the map $f^{p-1} : W \to \mathbb{D}_\rho$ has bounded distortion,

$$\operatorname{dist}(f(0), \partial W) \asymp \rho |Df^{p-1}(f(0))|^{-1}.$$

Hence for some $\eta > 0$,

$$\operatorname{dist}(f(0), \partial W) \ge \eta\rho |f(0) - 2|.$$

It follows that for $y \notin U'$ we have: $2|y|^2 \ge \eta\rho|f(0) - 2|$. On the other hand, since $|f(0) - 2| \to 0$ as $p \to \infty$, we have: $\eta\rho > |f(0) - 2|^{\kappa/4}$ for $p > p_0(\kappa, \rho)$. Hence $2|y|^2 \ge |f(0) - 2|^{1+\kappa/4}$. It implies by an elementary calculation that

$$|y|^{2-\kappa} \ge 2|y|^2 + (2|y|^2)^{(1+\kappa/4)^{-1}} \ge |f(y) - f(0)| + |f(0) - 2| \ge |f(y) - 2|,$$

provided $0 < \kappa \le 1/10$ and $|y| < e^{-\kappa^{-2}}$. □

Lemma 4.4. *For every* $\epsilon > 0$, $0 < \rho < \rho_0(\epsilon)$, *and for any period* $p \ge p_0(\epsilon, \rho)$, *the following property holds. Assume that* $y \in A'$ *and let* $m \ge 2$ *be the minimal moment such that* $|f^m(y) + 2| > 1/10$. *Then*

$$|Df^m(y)| \ge (2 - \epsilon)^m.$$

Proof. A simple consideration of the local dynamics near -2 shows that

$$|Df^{m-1}(f(y))| \asymp |f(y) - 2|^{-1}. \tag{4.3}$$

Hence $m \le K - \log|f_p(y) - 2|/\log\eta_p$, where $\eta_p = |Df(-2)|$. Since $\eta_p \to 4$ as $p \to \infty$, we have

$$(2-\epsilon)^m \le (2-\epsilon)^{K - \frac{\log|f(y)-2|}{\log\eta_p}} \le 2^K |f(y) - 2|^{\frac{-1+\kappa}{2}}$$

for $0 < \kappa < \kappa(\epsilon)$ and $p > p_0(\epsilon)$.

Set $\rho = e^{-\kappa^{-2}}$. By Lemma 4.3, if $p > p_0(\rho)$ then $y \in V' \setminus U'$ implies (4.2). On the other hand, (4.3) and (4.2) yields:

$$|Df^m(y)| = |Df(y)|\,|Df^{m-1}(f(y))| > K^{-1}|y|\,|f(y) - 2|^{-1} \ge K^{-1}|f(y) - 2|^{\frac{1}{2-\kappa} - 1}.$$

Thus, we just have to check

$$K^{-1}|f(y)-2|^{\frac{1}{2-\kappa}-1} \geq 2^K |f(y)-2|^{\frac{-1+\kappa}{2}},$$

that is,

$$|f(y)-2|^{\frac{\kappa(1-\kappa)}{4-2\kappa}} \leq \frac{1}{K2^K},$$

which follows from (4.2) and $|y| < \rho = e^{-\kappa^{-2}}$, provided κ is small enough. ☐

Note that we have obtained the same lower bound $(\log 2 - \epsilon)$ for the Lyapunov exponents of orbits that stay away from 0 and for those that originate quite near 0. It is because the multiplier of the postcritical fixed point -2 is big $(2^2 - \epsilon)$.

Lemma 4.5. *For every* $\epsilon > 0$, $0 < \rho < \rho_0(\epsilon)$, *and* $p > p_0(\epsilon, \rho)$, *we have*

(1) *If* $(x_0, \ldots, x_k) \in (A \underset{U\setminus V'}{\longleftarrow} U\setminus U')$ *then* $|Df^k(x_0)| \geq K(2-\epsilon)^k$, *where* $K = K(p, \rho) \to \infty$ *as* $p \to \infty$;

(2) *If* $(x_0, \ldots, x_k) \in (V' \underset{U\setminus V'}{\longleftarrow} U\setminus U')$ *then* $|Df^k(x_0)| \geq K(2-\epsilon)^k$ *for some absolute* K.

Proof. Let us deal with the first statement. Notice that since $\mathrm{mod}(\mathbb{V}\setminus\overline{\mathbb{U}})$ is large when p is large, for fixed ρ we also have $\lim_{p\to\infty} \mathrm{mod}(V\setminus\overline{U}) = \infty$. Since $\mathrm{mod}(U\setminus J(f)) \geq \mathrm{mod}(V\setminus\overline{U})$, we see that the distance $M(p, \rho)$ between ∂U and 0 satisfies $\lim_{p\to\infty} M(p, \rho) = \infty$. Since $x_k \in A$, we have $|x_k| \geq M(p, \rho)$. If $x_0 \notin V'$ then Lemma 4.2 shows that $|Df^k(x_0)| \geq K(p, \rho)(2-\epsilon)^k$, where $\lim_{p\to\infty} K(p, \rho) = \infty$. If $x_0 \in A'$, we let $2 \leq k_0 \leq k$ be minimal with $|f^{k_0}(x_0) + 2| > 1/10$. Then by Lemma 4.2, $|Df^{k-k_0}(f^{k_0}(x_0))| \geq K(p, \rho)(2-\epsilon)^{k-k_0}$, where $\lim_{p\to\infty} K(p, \rho) = \infty$, and by Lemma 4.4, $|Df^{k_0}(x_0)| \geq (2-\epsilon)^{k_0}$, so $|Df^k(x_0)| \geq K(p, \rho)(2-\epsilon)^k$, and the first statement follows.;

The second statement is analogous. ☐

Lemma 4.6. *Let* $\delta > 1$. *Then*

$$\lim_{p\to\infty} \sup \Xi_\delta(A \underset{U\setminus V'}{\longleftarrow} U\setminus U') = 0, \quad 0 < \rho < \rho_0(\delta),$$

$$\lim_{\rho\to 0} \limsup_{p\to\infty} \sup \Xi_\delta(V' \overset{+}{\underset{U\setminus V'}{\longleftarrow}} A') = 0,$$

$$\limsup_{p\to\infty} \sup \Xi_\delta(V' \underset{U\setminus V'}{\longleftarrow} U\setminus V') \leq K \equiv K(\delta), \quad 0 < \rho < \rho_0(\delta).$$

Proof. By the first statement of Lemma 4.5, for every $x \in A$ we have

$$\Xi_\delta(A \underset{U\setminus V'}{\longleftarrow} U\setminus U')(x) \equiv \Xi_\delta(\mathcal{F})(x) \leq \sum_{k\geq 1} \sum_{(x_0,\ldots,x_k=x)\in\mathcal{F}} |Df^k(x_0)|^{-\delta}$$

$$\leq \sum_{k\geq 1} 2^k K^{-\delta}(2-\epsilon)^{-\delta k} = K^{-\delta} \sum_{k\geq 1} \left(\frac{2}{(2-\epsilon)^\delta}\right)^k,$$

where $K = K(p, \rho)$, $\epsilon = \epsilon(p, \rho)$ satisfy $\lim_{p\to\infty} K(p, \rho) = \infty$, $\lim_{\rho\to 0}\lim_{p\to\infty} \epsilon(p, \rho) = 0$. The first estimate follows.

Let $m = m(p, \rho)$ be the minimal return time from A' to V'. Then $\lim_{\rho\to 0} \liminf_{p\to\infty} m(p, \rho) = \infty$. By the second statement of Lemma 4.5, for every $x \in V'$, we have

$$\Xi_\delta(V' \underset{U\setminus V'}{\overset{+}{\longleftarrow}} A')(x) \equiv \Xi_\delta(\mathcal{F})(x) \le \sum_{k\ge m} \sum_{(x_0,\ldots,x_k=x)\in\mathcal{F}} |Df^k(x_0)|^{-\delta}$$

$$\le \sum_{k\ge m} 2^k K^{-\delta}(2-\epsilon)^{-\delta k} = K^{-\delta} \sum_{k\ge m} \left(\frac{2}{(2-\epsilon)^\delta}\right)^k,$$

where K is an absolute constant and $\epsilon = \epsilon(p, \rho)$ satisfies $\lim_{\rho\to 0} \lim_{p\to\infty} \epsilon(p, \rho) = 0$. This gives the second estimate.

By the second statement of Lemma 4.5, for every $x \in V'$ we have

$$\Xi_\delta(V' \underset{U\setminus V'}{\longleftarrow} U\setminus V')(x) \equiv \Xi_\delta(\mathcal{F})(x) \le \sum_{k\ge 1} \sum_{(x_0,\ldots,x_k=x)\in\mathcal{F}} |Df^k(x_0)|^{-\delta}$$

$$\le \sum_{k\ge 1} 2^k K^{-\delta}(2-\epsilon)^{-\delta k} = K^{-\delta} \sum_{k\ge 1} \left(\frac{2}{(2-\epsilon)^\delta}\right)^k,$$

where K is an absolute constant and $\epsilon = \epsilon(p, \rho)$ satisfies $\lim_{\rho\to 0} \lim_{p\to\infty} \epsilon(p, \rho) = 0$. This gives the last estimate. □

Proof of Theorem 1.2. Since any quadratic-like map with connected Julia set satisfies $\delta_{\mathrm{cr}} \ge 1$, we only have to show that for every $\delta > 1$ and for every p sufficiently large, $\delta_{\mathrm{cr}}(f) \le \delta$, where $f = f_{p,\rho}$ is some quadratic-like representative of f_p.

Fix some $\delta > 1$. By Lemma 4.6, we can choose $\rho > 0$ so that

$$\limsup_{p\to\infty} \sup \Xi_\delta(V' \underset{U\setminus V'}{\overset{+}{\longleftarrow}} A') \le \frac{1}{4}.$$

Let P_δ be the quadratic polynomial defined in Lemma 3.1. Notice obvious inequalities

$$\max\{\Xi_\delta(A' \underset{U\setminus V'}{\longleftarrow} U\setminus V'),\ \Xi_\delta(U' \underset{U\setminus V'}{\longleftarrow} U\setminus V')\} \le \Xi_\delta(V' \underset{U\setminus V'}{\longleftarrow} U\setminus V'),$$

$$\max\{\Xi_\delta(A \underset{U\setminus V'}{\longleftarrow} U\setminus V'),\ \Xi_\delta(A \underset{U\setminus V'}{\longleftarrow} A')\} \le \Xi_\delta(A \underset{U\setminus V'}{\longleftarrow} U\setminus U'),$$

$$\max\{\Xi_\delta(A' \underset{U\setminus V'}{\overset{+}{\longleftarrow}} A'),\ \Xi_\delta(U' \underset{U\setminus V'}{\longleftarrow} A')\} \le \Xi_\delta(V' \underset{U\setminus V'}{\overset{+}{\longleftarrow}} A').$$

By Lemma 4.6, when p grows, the constant coefficient (3.4) of P_δ stays bounded, the linear coefficient (3.5) becomes smaller than $1/3$, and the quadratic term (3.6) goes to 0. In particular, for p large P_δ takes $[0, 2P_\delta(0)]$ into itself, and hence it has a fixed point. By Lemma 3.1, $\delta_{\mathrm{cr}}(f) \le \delta$ as desired. □

Corollary 4.7. *Let $F_p : z \mapsto z^2 + c_p$ be the straightening of f_p. Then*

$$\lim_{p\to\infty} \delta_{\mathrm{cr}}(F_p) = 1. \tag{4.4}$$

Proof. By Lemma 4.1, the germ f_p has a quadratic-like representative $f_p : U_p \to V_p$ with a big modulus: $\mathrm{mod}(V_p \setminus U_p) \to \infty$ as $p \to \infty$. Hence for p large, there is a quasi-conformal conjugacy between f_p and F_p with a small dilatation. This easily implies (see for instance Lemma 3.15 of [AL]) that f_p and F_p have close critical exponents: $\lim \delta_{\mathrm{cr}}(f_p) - \delta_{\mathrm{cr}}(F_p) = 0$. Together with Theorem 1.2, this implies that $\lim \delta_{\mathrm{cr}}(F_p) = \lim \delta_{\mathrm{cr}}(f_p) = 1$. □

5. Lebesgue measure of the Julia set. Below $f = f_p : U \to V$ will be the fixed point of nearly Chebyshev renormalization of period p, and $f' = f^p : U' \to V'$ will be its pre-renormalization as constructed in Lemma 4.1. Thus, $f'(z) = \lambda f(\lambda^{-1} z)$, where $\lambda = \lambda_p \in (-1, 0)$ is the scaling factor of f. We let as above $A = V \setminus U$, $A' = V' \setminus U'$. Furthermore, we let $U^k = \lambda^k U$, $V^k = \lambda^k V$, and $A^k = V^k \setminus U^k$.

We will need the following combinatorial lemma.

Lemma 5.1. *Let*

$$u^k = \sup \Xi_\delta(V^k \underset{U\setminus V^k}{\overset{+}{\longleftarrow}} A^k),$$

$$v_j^k = \sup \Xi_\delta^{[j]}(A^k \underset{U\setminus V^{k+1}}{\overset{+}{\longleftarrow}} A^k),$$

$$v^k = \lim_{j\to\infty} v_j^k = \sup \Xi_\delta(A^k \underset{U\setminus V^{k+1}}{\overset{+}{\longleftarrow}} A^k),$$

Then

$$u^{k+1} \le \sup \Xi_\delta(V' \underset{U\setminus V'}{\overset{+}{\longleftarrow}} A') + u^k(1+v^k) \sup \Xi_\delta(A \underset{U\setminus V'}{\longleftarrow} A') \\ (1 + \sup \Xi_\delta(V' \underset{U\setminus V'}{\longleftarrow} U \setminus V')), \tag{5.1}$$

$$v_{j+1}^k \le (1+v_j^k) u^k (1 + \sup \Xi_\delta(A \underset{U\setminus V'}{\longleftarrow} U \setminus V')). \tag{5.2}$$

Proof. Let $B^k = U\setminus(A^k \cup V^{k+1})$. Let us prove the first estimate.

We can decompose $V^{k+1} \xleftarrow[U\setminus V^{k+1}]{+} A^{k+1}$ into two groups:

$$V^{k+1} \xleftarrow[B^k]{+} A^{k+1},$$

which takes into account the orbits that do not land at the annulus A^k, and

$$V^{k+1} \xleftarrow[B^k]{} A^k \xleftarrow[U\setminus V^{k+1}]{} A^k \xleftarrow[B^k]{} A^{k+1},$$

which accounts for the orbits landing at A^k and marks the first and the last landings. Thus

$$\begin{aligned} u^{k+1} = \sup \Xi_\delta(V^{k+1} \xleftarrow[U\setminus V^{k+1}]{+} A^{k+1}) &\le \sup \Xi_\delta(V^{k+1} \xleftarrow[B^k]{+} A^{k+1}) \\ &+ (\sup \Xi_\delta(V^{k+1} \xleftarrow[B^k]{} A^k) \cdot \sup \Xi_\delta(A^k \xleftarrow[U\setminus V^{k+1}]{} A^k) \cdot \\ &\sup \Xi_\delta(A^k \xleftarrow[B^k]{} A^{k+1})). \end{aligned} \tag{5.3}$$

Notice that

$$\Xi_\delta(V^{k+1} \xleftarrow[B^k]{+} A^{k+1})(x) = \Xi_\delta(V' \xleftarrow[U\setminus V']{+} A')(\lambda^{-k}(x)), \tag{5.4}$$

$$\Xi_\delta(A^k \xleftarrow[B^k]{} A^{k+1})(x) = \Xi_\delta(A \xleftarrow[U\setminus V']{} A')(\lambda^{-k}x), \tag{5.5}$$

and

$$\sup \Xi_\delta(A^k \xleftarrow[U\setminus V^{k+1}]{} A^k) = 1 + \sup \Xi_\delta(A^k \xleftarrow[U\setminus V^{k+1}]{+} A^k) = 1 + v^k, \tag{5.6}$$

the 1 accounting for trivial orbits. Plugging (5.4)–(5.6) into (5.3) we get

$$u^{k+1} \le \sup \Xi_\delta(V' \xleftarrow[U\setminus V']{+} A') + (1+v^k) \sup \Xi_\delta(A \xleftarrow[U\setminus V']{} A') \sup \Xi_\delta(V^{k+1} \xleftarrow[B^k]{} A^k). \tag{5.7}$$

We can decompose $V^{k+1} \xleftarrow[B^k]{} A^k$ into two groups,

$$V^{k+1} \xleftarrow[U\setminus V^k]{} A^k, \quad \text{and} \quad V^{k+1} \xleftarrow[B^k]{} U^k\setminus V^{k+1} \xleftarrow[U\setminus V^k]{} A^k.$$

Thus

$$\begin{aligned} \sup \Xi_\delta(V^{k+1} \xleftarrow[B^k]{} A^k) \le \sup \Xi_\delta(V^{k+1} &\xleftarrow[U\setminus V^k]{} A^k) \\ &+ \sup \Xi_\delta(V^{k+1} \xleftarrow[B^k]{} U^k\setminus V^{k+1}) \\ &\sup \Xi_\delta(U^k\setminus V^{k+1} \xleftarrow[U\setminus V^k]{} A^k). \end{aligned} \tag{5.8}$$

Notice that

$$\Xi_\delta(V^{k+1} \underset{B^k}{\longleftarrow} U^k \setminus V^{k+1})(x) = \Xi_\delta(V' \underset{U\setminus V'}{\longleftarrow} U\setminus V')(\lambda^{-k}x), \tag{5.9}$$

$$\max\{\sup \Xi_\delta(V^{k+1} \underset{U\setminus V^k}{\longleftarrow} A^k), \sup \Xi_\delta(U^k \setminus V^{k+1} \underset{U\setminus V^k}{\longleftarrow} A^k), \sup \Xi_\delta(A^k \overset{+}{\underset{U\setminus V^k}{\longleftarrow}} A^k)\}$$

$$= \sup \Xi_\delta(V^k \overset{+}{\underset{U\setminus V^k}{\longleftarrow}} + A^k) = u^k. \tag{5.10}$$

Plugging (5.9) and (5.10) into (5.8), and plugging the resulting expression for $\sup \Xi_\delta(V^{k+1} \underset{B^k}{\longleftarrow} A^k)$ into (5.7) gives (5.1).

Let us prove the second estimate. We will omit the truncation parameter (j or $j+1$). We can rewrite $A^k \overset{+}{\underset{U\setminus V^{k+1}}{\longleftarrow}} A^k$ as $A^k \overset{+}{\underset{B^k}{\longleftarrow}} A^k \underset{U\setminus V^{k+1}}{\longleftarrow} A^k$. Thus

$$\sup \Xi_\delta(A^k \overset{+}{\underset{U\setminus V^{k+1}}{\longleftarrow}} A^k) \le \sup \Xi_\delta(A^k \overset{+}{\underset{B^k}{\longleftarrow}} A^k) \sup \Xi_\delta(A^k \underset{U\setminus V^{k+1}}{\longleftarrow} A^k). \tag{5.11}$$

Plugging (5.6) into (5.11) we get

$$v^k \le (1+v^k) \sup \Xi_\delta(A^k \overset{+}{\underset{B^k}{\longleftarrow}} A^k). \tag{5.12}$$

We can split $A^k \overset{+}{\underset{B^k}{\longleftarrow}} A^k$ into two groups: $A^k \overset{+}{\underset{U\setminus V^k}{\longleftarrow}} A^k$ and $A^k \underset{B^k}{\longleftarrow} U^k \setminus V^{k+1} \underset{U\setminus V^k}{\longleftarrow} A^k$. Thus

$$\sup \Xi_\delta(A^k \overset{+}{\underset{B^k}{\longleftarrow}} A^k) \le \sup \Xi_\delta(A^k \overset{+}{\underset{U\setminus V^k}{\longleftarrow}} A^k) + \sup \Xi_\delta(A^k \underset{B^k}{\longleftarrow} U^k \setminus V^{k+1}) \tag{5.13}$$

Notice that

$$\Xi_\delta(A^k \underset{B^k}{\longleftarrow} U^k \setminus V^{k+1})(x) = \Xi_\delta(A \underset{U\setminus V'}{\longleftarrow} U\setminus V')(\lambda^{-k}x), \tag{5.14}$$

Plugging (5.14) and (5.10) into (5.13) we get

$$\sup \Xi_\delta(A^k \overset{+}{\underset{B^k}{\longleftarrow}} A^k) \le u^k + u^k \sup \Xi_\delta(A \underset{U\setminus V'}{\longleftarrow} U\setminus V'). \tag{5.15}$$

Plugging (5.15) into (5.12) gives (5.2). □

Proof of Theorem 1.3. By Lemma 4.6, there exists $K \equiv K(2) > 0$ such that if one takes ρ sufficiently small, then for all p sufficiently large we have

$$\sup \Xi_2(V' \underset{U\setminus V'}{\overset{+}{\longleftarrow}} A') < \frac{1}{100}, \tag{5.16}$$

$$\sup \Xi_2(A \underset{U\setminus V'}{\longleftarrow} A') < \frac{1}{5K+5}, \tag{5.17}$$

$$\sup \Xi_2(V' \underset{U\setminus V'}{\longleftarrow} U\setminus V') < 2K, \tag{5.18}$$

$$\sup \Xi_2(A \underset{U\setminus V'}{\longleftarrow} U\setminus V') < \frac{1}{100}. \tag{5.19}$$

Let us show by induction that for every $k \geq 0$ we have

$$u^k \leq \frac{1}{10}, \tag{5.20}$$

where u^k is as in Lemma 5.1. Notice that $u^0 = 0$, so (5.20) holds for $k = 0$. Assuming that (5.20) holds for some k, notice that (5.2) and (5.19) imply

$$v^k_{j+1} \leq \frac{1}{5}(1 + v^k_j),$$

for $j \geq -1$. Since $v^k_{-1} = 0$, this implies by induction that $v^k_j \leq \frac{1}{4}$ for every $j \geq -1$, so $v^k = \lim_{j\to\infty} v^k_j \leq \frac{1}{4}$. By (5.1) and (5.16)–(5.18), we have

$$u^{k+1} \leq \frac{1}{100} + \frac{1}{10}\frac{5}{4}\frac{1}{5K+5}(2K+1) \leq \frac{1}{10}.$$

By induction, (5.20) holds for all $k \geq 0$.

Let

$$X_k = \bigcup_{r\geq 1} f^{-r} V^k.$$

Then

$$\text{area}(X_k \cap A^k) = \int_{A^k} 1_{X^k} dx = \int_{V^k} \Xi_2(V^k \underset{U\setminus V^k}{\overset{+}{\longleftarrow}} A^k) dx \leq \int_{V^k} u^k dx \leq \frac{1}{10}\text{area}(V^k).$$

Notice that $X^k \cap V^k = U^k \cup (X^k \cap A^k)$. Thus,

$$\frac{\text{area}(X^k \cap V^k)}{\text{area} V^k} \leq \frac{1}{10} + \frac{\text{area} U^k}{\text{area} V^k} \leq \frac{1}{5}, \tag{5.21}$$

where we have used that area $U^k \leq \frac{1}{10}\text{area} V^k$, which holds since $\text{mod}(V^k \setminus \overline{U^k}) = \text{mod } A$ is big for large p (by Lemma 4.1).

The conclusion of the argument is standard. Let

$$X = \{x \in J(f),\ 0 \in \omega(x)\}.$$

Notice that X is fully invariant: $X = f^{-1}(X) = f(X)$. By [L1], for almost every $x \in J(f)$, $\omega(x) \subset \omega(0)$. Since $\omega(0)$ is a minimal set containing 0, we conclude that $\operatorname{area} X = \operatorname{area} J(f)$. Let us show that $\operatorname{area} X = 0$.

Assume that this is not the case. By the Lebesgue Density Points Theorem, there exists a density point $x \in X$. Let $r_k \geq 0$ be minimal such that $f^{r_k}(x) \in V^k$. We may assume that x is not a preimage of 0, so that $r_k \to \infty$. Let W^k be the connected component of $f^{-r_k}(V^k)$ containing x. Then $f^{r_k} : W^k \to V^k$ admits a univalent extension onto $\mathbb{V}^k \equiv \lambda^k \mathbb{V}$, and since $\operatorname{mod}(\mathbb{V}^k \setminus \overline{V^k})$ is big, it has distortion bounded by 2. It also follows that W^k contains a round disk of radius $\frac{1}{10}\operatorname{diam}\,(W^k)$. Since $r_k \to \infty$ and $W^k \subset f^{-r_k}(V)$, $\limsup W^k \subset K(f)$. Since $K(f)$ has empty interior, we conclude that $\operatorname{diam}\,(W^k) \to 0$. Notice that

$$\frac{\operatorname{area}(V^k \setminus X)}{\operatorname{area} V^k} \leq 10 \frac{\operatorname{area}(W^k \setminus X^k)}{\operatorname{area} W^k} \leq 1000 \frac{\operatorname{area}(\mathbb{D}_{\operatorname{diam}\,(W^k)}(x) \setminus X)}{\operatorname{area}(\mathbb{D}_{\operatorname{diam}\,(W^k)}(x))},$$

and since x is a density point of X, we have

$$\frac{\operatorname{area}(V^k \setminus X)}{\operatorname{area}(V^k)} \to 0. \tag{5.22}$$

Obviously, $X \subset X_k$, so (5.22) and (5.21) give the desired contradiction. □

Acknowledgement. The authors thank the referee for carefully reading the manuscript. This work was partly supported by the the Clay Mathematics Institute, the Guggenheim Foundation, NSF and NSERC. It was carried out during authors' meetings at the IMS at Stony Brook, at the University of Toronto, and the Institut Henri Poincaré. The authors thank all these Institutions and Foundations for hospitality and support.

References

[AL] A. Avila and M. Lyubich, Hausdorff dimension and conformal measures of Feigenbaum Julia sets. Preprint IMS at Stony Brook, # 5, 2004.

[B] C. J. Bishop, Minkowski dimension and the Poincaré exponent. Michigan Math. J. 43 (1996), no. 2, 231–246.

[DH] A. Douady and J. H. Hubbard, On the dynamics of polynomial-like maps. Ann. Sc. Éc. Norm. Sup., v. 18 (1985), 287-343.

[L1] M. Lyubich, Typical behavior of trajectories of a rational mapping of the sphere. Dokl. Akad. Nauk SSSR, v. 268 (1982), 29–32.

[L2] M. Lyubich, Dynamics of quadratic polynomials, I-II. Acta Math., v. 178 (1997), 185-297.

[Mc1] C. McMullen, Complex dynamics and renormalization. Annals of Math. Studies, v. 142, Princeton University Press, 1994.

[Mc2] C. McMullen, Renormalization and 3-manifolds which fiber over the circle. Annals of Math. Studies, v. 135, Princeton University Press, 1996.

[MS] W. de Melo and S. van Strien, One-dimensional dynamics. Springer, 1993.

[S1] D. Sullivan, Conformal dynamical systems. Geometric dynamics (Rio de Janeiro, 1981), 725–752, Lecture Notes in Math., 1007, Springer, Berlin, 1983.

[S2] D. Sullivan, Bounds, quadratic differentials, and renormalization conjectures. AMS Centennial Publications. 2: Mathematics into Twenty-first Century (1992).

[Y] B. Yarrington, Local connectivity and Lebesgue measure of polynomial Julia sets. Thesis, Stony Brook 1995.

Laboratoire de Probabilités et Modèles aléatoires
Université Pierre et Marie Curie–Boîte courrier 188
75252–Paris Cedex 05,
France
e-mail: artur@ccr.jussieu.fr

Mathematics Department and IMS,
SUNY Stony Brook,
Stony Brook,
NY 11794, USA
e-mail: mlyubich@math.sunysb.edu

Department of Mathematics,
University of Toronto,
Ontario,
Canada M5S 3G3
e-mail: misha@math.toronto.edu

Parabolic explosion and the size of Siegel disks in the quadratic family

Arnaud Chéritat

Abstract. For an irrational number θ, let $r(\theta)$ be the conformal radius at $z=0$ of the polynomial $z \mapsto e^{i2\pi\theta}z + z^2$. Let Y be Yoccoz's arithmetic Brjuno function. The linearization theorem of Yoccoz implies that there exists a real constant C_1 such that for all θ irrational with $Y(\theta) < +\infty$,

$$C_1 < Y(\theta) + \log(r(\theta)).$$

Yoccoz also proved a similar but weaker inequality on the other side, which in particular implies non linearizability of the polynomial when $Y(\theta) = +\infty$. The author found a new and independent proof of non linearizability. One of the tools of this new proof is the *control on parabolic explosion*. In a joint work with X. Buff, we pushed this further and were able to prove the existence of a real constant C_2 such that for all θ with $r(\theta) > 0$,

$$Y(\theta) + \log(r(\theta)) < C_2,$$

enhancing Yoccoz's second inequality.

1.1. Notations. Let

$$P_\theta(z) = e^{i2\pi\theta}z + z^2$$

The point $z=0$ is fixed with multiplier $e^{i2\pi\theta}$. For $\theta \in \mathbb{R}$, it is an indifferent fixed point, and it is the only non repelling cycle. If $z=0$ is linearizable,

- we note $\Delta(\theta)$ the Siegel disk of P,
- we note $r(\theta)$ its conformal radius w.r.t. $z=0$.

The conformal radius can be defined as $\phi'(0)$ where $\phi : \mathbb{D} \to \Delta(\theta)$ is the unique conformal map sending 0 to 0 with real positive derivative. If $z=0$ is not linearizable, we define $\Delta(\theta) = \varnothing$ and $r(\theta) = 0$.

The Yoccoz function (that he calls the Brjuno function)

$$Y : \mathbb{R} \to \,]0, \ +\infty]$$

is defined on rationals by

$$Y(p/q) = +\infty.$$

For $\theta \in \mathbb{R}$ irrational,

$$Y(\theta) = \sum_{n=0}^{+\infty} \theta_0 \cdots \theta_{n-1} \log \frac{1}{\theta_n},$$

where θ_n is defined inductively by $\theta_0 = \mathrm{frac}(\theta) = \theta - \lfloor \theta \rfloor \in]0, 1[$ and $\theta_{n+1} = \mathrm{frac}(1/\theta_\mathrm{n})$ (this is the Gauss algorithm).

The Brjuno sum of a rational number is defined by

$$B(p/q) = +\infty.$$

The Brjuno sum of an irrational θ belongs to $]0, +\infty]$, and is defined by

$$B(\theta) = \sum_{n=0}^{+\infty} \frac{\log q_{n+1}}{q_n}$$

where p_n/q_n are the convergents of the continued fraction expansion of θ.

The set of Brjuno numbers is defined as

$$\mathcal{B} = \{\theta \in \mathbb{R} \mid B(\theta) < +\infty\}.$$

The Yoccoz function and the Brjuno sum are equivalent in the following sense:

$$B(\theta) = +\infty \Longleftrightarrow Y(\theta) = +\infty$$

$$|B(\theta) - Y(\theta)| \text{ is bounded on } \mathcal{B}$$

2. Short and incomplete historical background. Let f be a holomorphic germ at 0, fixing 0 with multiplier $e^{i2\pi\theta}$.

Theorem (Siegel, 1942)**.** *If θ is Diophantine, then f is linearizable.*

Theorem (Brjuno, 1965)**.** *If $\theta \in \mathcal{B}$, then f is linearizable.*

Theorem (Yoccoz, 1988)**.** $(\exists C > 0)$ $(\forall \theta \in \mathcal{B})$*, if $f : \mathbb{D} \to \mathbb{C}$ is univalent and fixes* 0 *with multiplier $e^{i2\pi\theta}$, then*

$$B(0, e^{-Y(\theta)-C}) \subset \Delta(f).$$

Since the restriction of P_θ to $B(0, 1/2)$ is injective, the previous theorem implies $\log r(\theta) \geq -Y(\theta) - C - \log 2$, i.e.,

$$-C - \log 2 \leq \log r(\theta) + Y(\theta).$$

Theorem (Yoccoz, germs)**.** $(\exists C' > 0)$ $(\forall \theta \in \mathbb{R})$*, there exists an $f : \mathbb{D} \to \mathbb{C}$ univalent and fixing* 0 *with multiplier $e^{i2\pi\theta}$, with*

$$\log r(f) \leq -Y(\theta) + C'$$

if $Y(\theta)$ is finite, and f not linearizable if $Y(\theta) = +\infty$.

With a quasiconformal surgery procedure, and ideas of Ill'Yashenko, Yoccoz then transforms these functions $f : \mathbb{D} \to \mathbb{C}$ into quadratic maps.

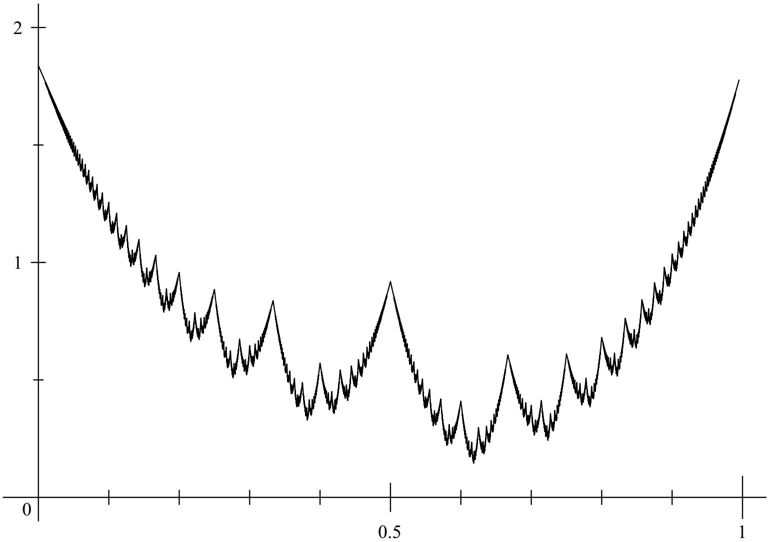


Figure 1: The function $\theta \mapsto \log(r(\theta)) + Y(\theta)$

Theorem (Yoccoz, quadratic). $(\forall \varepsilon > 0)$, $(\exists C(\varepsilon) \in \mathbb{R})$ $(\forall \theta \in \mathbb{R})$,

1. *if* $Y(\theta) = +\infty$, P_θ *is not linearizable,*
2. *if* $Y(\theta)$ *is finite,*

$$\log r(\theta) \leq -(1-\varepsilon)Y(\theta) + C(\varepsilon).$$

As a Corollary,

$$-C \leq \log\, r(\theta) + Y(\theta) \leq \varepsilon Y(\theta) + C(\varepsilon).$$

It was conjectured that one can get rid of the ε.

Conjecture. The function $\theta \in \mathcal{B} \mapsto \log(r(\theta)) + Y(\theta)$ is bounded.

In 1988, Stefano Marmi made computer experiments, showing that this function seems to be the restriction to $\mathcal{B}$ of a continuous function on $\mathbb{R}$. Later, there is the

Conjecture (Marmi, Moussa, Yoccoz)**.** The function $\theta \in \mathcal{B} \mapsto \log(r(\theta)) + Y(\theta)$ is 1/2-Hölder.

3. Our contribution.

Theorem 1 ([BC2])**.** *The function* $\theta \in \mathcal{B} \mapsto \log(r(\theta)) + Y(\theta)$ *is bounded from above.*

The lower bound following from Yoccoz 88, this proves the first conjecture.

The proof relies on three ingredients

1. the control on parabolic explosion (C. 1999)
2. estimating how cycles block each other via the conformal radius (C. 2001)
3. the relative Schwarz lemma (Buff, 2003)

With the first two point, the author was able to reprove point 1 of the Yoccoz quadratic theorem (see [C2]).

3.1. Parabolic explosion. For all irreducible p/q (meaning $q>0$), we have

$$P^q_{p/q}(z) = z + Cz^{q+1} + O(z^{q+2}),$$

and the fact that the polynomials P_θ have only one critical point implies that $C \neq 0$. Therefore, $P^q_{p/q}$ has $q+1$ fixed points at $z=0$. When we perturb p/q into θ (complex values of θ allowed), these points explode into a cycle of P_θ of length q (to which we must add the common fixed point $z=0$). According to the implicit function theorem, the periodic points move holomorphically w.r.t. the parameter, as long as their multiplier is different from 1, which is equivalent to asking that they do not collide with another periodic point of dividing period. This happens at $\theta = p/q$. The monodromy around this value in fact permutes the points of the cycle. The following elementary proposition gives a clearer view of what happens.

Let

$$\mathcal{S}_q = \{\theta \in \mathbb{C} \mid P^q_\theta(z) - z \text{ has a no multiple root}\}.$$

$$R_{p/q} = \max\{R > 0 \mid B(p/q, R^q) - \{p/q\} \subset \mathcal{S}_q\}$$

Note that $R_{p/q} \leq 1$.

Proposition 1 ([C1]). *$\exists \chi = \chi_{p/q} : B(0, R_{p/q}) \to \mathbb{C}$, holomorphic, such that*

- $\chi(0) = 0$
- $\forall \delta \in \mathrm{Def}(\chi)$ *with* $\delta \neq 0$, $\chi(\delta)$ *is a parabolic point of* P_θ *of period* q, *with*

$$\theta = \frac{p}{q} + \delta^q.$$

The q different values of δ associated to a same $\theta \in B(p/q, R^q_{p/q}) - \{p/q\}$ map by χ to a whole cycle of P_θ, of length q.

Thus, taking the parameter δ instead of θ cancels the monodromy.

Controlling the parabolic explosion means controlling the functions $\chi_{p/q}$. From this point of view, the following lemma is fundamental:

Lemma 1 (Douady, [C1]). $(\exists K > 0)$ $\forall p/q$ *irreducible*,

$$R^q_{p/q} \geq \frac{K}{q^3}.$$

This is a corollary of the Yoccoz inequality on the limbs of the Mandelbrot set, combinatorics of degree 2 polynomials, and Pythagoras' theorem. In our applications, the exact exponent of the power in K/q^3 is not important. It is conjectured that the best possible is K/q^2.

With just that, one can already prove independently from Yoccoz 88 (see [C1]) that

$$\frac{\log q_{n+1}}{q_n} \not\longrightarrow 0 \implies P_\theta \text{ is not linearizable.}$$

The proof works like this: let p_n/q_n be the convergents of the continued fraction expansion of θ. We want to apply the Schwarz lemma to χ_{p_n/q_n}. First, note that Douady's lemma implies

$$R_{p_n/q_n} \underset{n\to+\infty}{\longrightarrow} 1.$$

Then, for all $n \in \mathbb{N}$, let δ_n be any of the q_n complex numbers such that

$$\theta = p_n/q_n + \delta_n^{q_n}.$$

Elementary properties of the convergents imply

$$\log |\delta_n| = -\frac{\log q_{n+1}}{q_n} + o(1)$$

as $n \longrightarrow +\infty$. Therefore,

$$\limsup \frac{|\delta_n|}{R_{p_n/q_n}} < 1.$$

The functions $\chi_{p/q}$ all take value in some same ball $B(0, M_0)$ (the values are periodic points of P_θ with $\mathrm{Im}(\theta) \leq 1$). Therefore they form a normal family. Any limit of χ_{p_n/q_n} is defined on $\mathbb{D}$, is analytic, maps 0 to 0, and $\mathbb{D}$ in the Julia set. If $r(\theta) = 0$, J has empty interior and the limit is constant $= 0$. If $r(\theta) > 0$, the limit maps *in* the Siegel disk. Now, if P_θ had a Siegel disk, then for n big enough, the Schwarz lemma would imply that $\chi_{p_n/q_n}(\delta_n)$ is a periodic point of P_θ that lies in the Siegel disk (and is not equal to 0, since $\delta_n \neq 0$), which is impossible.

Along the same lines, still assuming $\frac{\log q_{n+1}}{q_n} \not\to 0$, one can then prove that P_θ has small cycles (meaning that every neighborhood of 0 contains a whole cycle of P_θ, apart from $z = 0$).

3.2. Blocking. The previous argument works for irrationals whose Brjuno summand does not tend to 0. To extend the proof to numbers whose Brjuno sum diverges but with a summand tending to 0, we need a way to make the sum of the $\log |\delta_n|$ appear in the bound: recall that $\log |\delta_n| = -\frac{1}{q_n} \log q_{n+1} + o(1)$, whence divergence of the Brjuno sum implies divergence of the sum of $\log |\delta_n|$. So

Question: why do the $\log|\delta_n|$ accumulate?

Answer: because the previously exploded cycles (those from $p_0/q_0, \ldots, p_n/q_n$) block the explosion of the next one.

Indeed, by assumption on $R_{p/q}$, the periodic point $\chi_{p/q}(\delta)$ does not collide with a cycle of period $< q$.

To make this work, the author used three main ideas:

α. Measure the hindrance of present cycles via r_n, the conformal radius at 0 of the following set: $\mathbb{C}$ minus the cycles of P_θ coming from $p_0/q_0, \ldots, p_n/q_n$. To ensure non linearizability, it is enough to prove that $r_n \longrightarrow 0$ as $n \longrightarrow +\infty$. The loss of conformal radius will be measured via $-(\log r_{n+1} - \log r_n) > 0$, and compared to $-\log|\delta_n|$.

From r_n to r_{n+1}, there is a new cycle. It comes from the explosion of the parabolic point of $P_{p_{n+1}/q_{n+1}}$. We said that the points of this cycle can not collide with a cycle of lower period. Therefore, the explosion takes place in the complement of the previously exploded cycles. Unfortunately, that set is not still: during the explosion, the parameter varies, and thus the previous cycles move. More precisely, let $p = p_{n+1}$ and $q = q_{n+1}$; when δ varies, $\theta' = p/q + \delta^q$ varies too, so the cycles that the function $\chi_{p/q}$ avoids, also move.

β. To reduce to a still target set, we take a holomorphically varying universal cover (mapping 0 to 0 with derivative 1) of $\mathbb{C}$ minus the previous cycles. That is, to each θ' we associate a simply connected open set $U_{\theta'}$ and a analytic universal cover $\xi_\theta : U_\theta \to \mathbb{C}$ minus the previous cycles, which is a universal cover. The set $U_{\theta'}$ is taken to be the round disk $B(0, r_n)$ when $\theta' = 0$. Since the movement of θ' is much slower than the movement of δ, the varying universal cover remains close to the disk $B(0, r_n)$, during the explosion of p/q. In particular, it remains contained in a disk of radius slightly bigger than r_n.

This moving universal cover is constructed by using holomorphic motions, Slodkowski's theorem and Ahlfors-Bers straightening of Beltrami forms.

Since the function $\delta \mapsto \chi_{p/q}(\delta)$ that follows the explosion in terms of the q-th root δ of $\theta - p/q$ avoids the previous cycles and is defined on a simply connected set (a disk), it is possible to look a the explosion in the new coordinates given by the varying universal cover. This yields well defined a branch of $\delta \mapsto \xi^{-1}_{\delta^q + p/q} \circ \chi_{p/q}\,(\delta)$ for $\delta \in B(0, R_{p/q})$.

The logarithmic loss of conformal radius $-(\log r_{n+1} - \log r_n) > 0$ is equal to the logarithmic loss of conformal radius of $B(0, r_n)$ when we remove the image by this branch of all the q-th roots of $\theta - p/q$.

Now the branch takes value in a disk of radius r'_n slightly bigger than r_n. So we will bound the logarithmic loss $-(\log r_{n+1} - \log r_n)$ from below by $\log(r_n/r'_n)$ plus the logarithmic loss of conformal radius of this bigger disk, when we remove from it the image of a holomorphic function from $B(0, R_{p/q})$ to the bigger disk mapping 0 to 0, taken at the q-th roots of $\theta - p/q$.

γ. The latter loss is shown to be *comparable* to the loss of the toy model. The toy model is the following: we consider $B(0, R_{p/q})$ and remove from it the q-th roots of $\theta - p/q$.

In this step, comparable means the author was able to show: the loss is less than the toy model's loss divided by a universal constant.

Now, for the toy model, this loss is easily shown to be of the form

$$\log \frac{1}{|\delta_n|} - \text{rest}$$

where the rest is small.

Indeed, $B(0, R_{p/q})$ is close to $\mathbb{D}$, and the distance of the q-th roots of $\theta - p/q$ to the boundary of $\mathbb{D}$ is equal to $1 - |\delta_n|$, which is small. The distance between two consecutive roots is much smaller. So these points act like a barrier, and the conformal radius is almost equal to $|\delta_n|$.

This was enough to obtain a new and independent proof of non-linearizability when the Brjuno sum diverges.

3.3. The relative Schwarz lemma. When bounding the logarithmic loss $-(\log r_{n+1} - \log r_n)$ from below by $\frac{1}{q_n} \log q_{n+1}$, we had several error terms. We compared $\frac{1}{q_n} \log q_{n+1}$ to $\log |\delta_n|$. We compared $\log r_n$ to $\log r'_n$. We compared the toy model's loss to $\log |\delta_n|$. But in these three cases, the error term is bounded from above by a universal sequence, whose sum converges. The only point which prevented from obtaining theorem 1 was step γ: that some log-loss is comparable to the log-loss for the toy model in the sense that the quotient is bounded from above; there is a bounded multiplicative error term; since the log-loss of the toy model may be big, the additive error term may be also big.

X. Buff enhanced step γ by proving that the loss for an arbitrary f is *at least* the loss of the toy model. So in fact, there is *no error term* in the upper bound of step γ. (It is even better than having a universal and summable error term.) The other error terms remain.

This yielded Theorem 1. To be precise, we proved in [BC2] the following inequality:

$$\log \rho_n + \sum_{k=0}^{n} \frac{\log q_{n+1}}{q_n} < C$$

where C is a universal constant and ρ_n is the following variant of r_n: it is the conformal radius at 0 of the complement in $\mathbb{C}$ of the external ray of argument 0 and all the cycles of period $\leq q_n$ (except 0).

So now, let us go into this relative Schwarz lemma. For a Riemann surface X, we note ρ_X its hyperbolic metrics: this is a conformal metrics with curvature equal to -1 everywhere. If it exists, it is unique and X is said to be hyperbolic.

Relative Schwarz lemma (Buff, 2003)**.** *Let X, Y be hyperbolic Riemann surfaces, and $f : X \to Y$ an analytic map. Let $Y' \subset Y$ be open. Let $X' = f^{-1}(Y')$. Then,*

$$\frac{f^*\rho_Y}{\rho_X} \leq \frac{f^*\rho_{Y'}}{\rho_{X'}}.$$

The proof uses Ahlfors' ultrahyperbolic metrics (this is McMullen who suggested us to try them). It can be found in [BC2].

The inequality of the relative Schwarz lemma can be read in two ways. First, $\frac{f^*\rho_Y}{\rho_X}$, which is a function over X, measures the expansion of f in terms of hyperbolic metrics. Since analytic maps always contract the hyperbolic metrics (this follows from the classical Schwarz inequality), this number is ≤ 1. So the lemma states that f is less contracting from X' to Y' than from X to Y. This is where the relative Schwarz lemma's name comes from.

Second, let us rewrite the lemma as

$$\frac{\rho_{X'}}{\rho_X} \leq \frac{f^*\rho_{Y'}}{f^*\rho_Y} = \frac{\rho_{Y'}}{\rho_Y} \circ f.$$

These two functions, defined on X', are now ≥ 1 (as also follows from the classical Schwarz inequality). The relative Schwarz lemma states that the hyperbolic metrics of subsets varies less for preimages.

The second point of view can be restated in terms of conformal radius: if $X \subset \mathbb{C}$, let $\rho_X(z) = \rho(z)|dz|$. The conformal radius of X at z is equal to $\mathrm{rad}(X, z) = 1/\rho(z)$. Then the relative Schwarz lemma reads

$$\frac{\mathrm{rad}(Y', f(z))}{\mathrm{rad}(Y, f(z))} \leq \frac{\mathrm{rad}(X', z)}{\mathrm{rad}(X, z)}.$$

In the application to the proof of theorem 1, we take $X = B(0, R_{p,q})$, $Y = B(0, r'_n)$, $Y' = Y \backslash f(Z)$ where Z is the set of q-th roots of $\theta - p/q$, and $f(\delta) = \xi^{-1}_{\delta^q + p/q} \circ \chi_{p/q}(\delta)$.

References

[BC1] X. Buff, A. Chéritat, *Quadratic Siegel disks with smooth boundaries*, preprint, n°242 at Université Paul Sabatier, Toulouse, France (2002).

[BC2] X. Buff, A. Chéritat, *Upper bound for the size of Quadratic Siegel disks*, Inventiones Mathematicae 156, n°1, 1–24 (2004).

[C1] A. Chéritat, *Estimates on the speed of explosion of the parabolic fixed points of quadratic polynomials and applications*, preprint, n°99–77 at Université Paris-Sud, Orsay, France (1999).

[C2] A. Chéritat, *Recherche d'ensembles de Julia de mesure de Lebesgue positive*, Thesis at Université Paris-Sud, Orsay, France (2001).

Laboratoire de Mathématiques Emile Picard - UMR 5580
Institut de Mathématiques de Toulouse
Université Paul Sabatier Toulouse 3 - Bât 1R3
31 062 TOULOUSE cedex 9 France
e-mail: cheritat@picard.ups-tlse.fr

Sierpinski Carpets and Gaskets as Julia sets of Rational Maps

*Paul Blanchard**
*Robert L. Devaney**
*Daniel M. Look**
*Monica Moreno Rocha***
*Pradipta Seal**
*Stefan Siegmund****
*David Uminsky**

0. Introduction. In recent years, it has been shown that the family of rational maps arising from *singular perturbations* of the simple polynomials $z \mapsto z^n$ have some interesting properties from a dynamical systems as well as a topological perspective. In this paper we survey some of these results. In addition, we provide proofs of these results in several special and illustrative cases. While the cases we describe here are by no means the most general, they do serve to illustrate the types of techniques that can be used in the general cases.

By a singular perturbation of z^n, we mean a map of the form $z \mapsto z^n + \lambda/z^m$ where λ is a complex parameter. Of primary interest is the Julia set of these maps. From an analytic viewpoint, the Julia set is the set of points at which the family of iterates of the map fails to be a normal family in the sense of Montel. From a dynamics point of view, the Julia set is the set of points on which the map is chaotic.

As is well known, the Julia set of z^n for $n \geq 2$ is just the unit circle. When we add the term λ/z^m for $\lambda \in \mathbb{C}$, $\lambda \neq 0$, several things happen. First of all, the degree of the rational map suddenly increases from n to $n+m$. Secondly, the superattracting fixed point at the origin becomes a pole, while ∞ remains a superattracting fixed point. As a consequence, an open set around the origin now lies in the basin of attraction of ∞. In between this neighborhood of 0 and the basin at ∞, the Julia set undergoes a significant transformation.

For example, if $1/n + 1/m < 1$, McMullen [13] has shown that, when $|\lambda|$ is small, the Julia set explodes from a single circle to a Cantor set of simple closed curves surrounding the origin. See Figure 1. When n and m do not satisfy the McMullen condition, the situation is quite different. In [2] it is shown that, in the cases $n = m = 2$ or $n = 2$, $m = 1$, there are infinitely many open sets of λ-values in any neighborhood of

* Boston University, Boston, MA 02215, USA
** Tufts University, Medford, MA, 02155 USA
*** University of Frankfurt

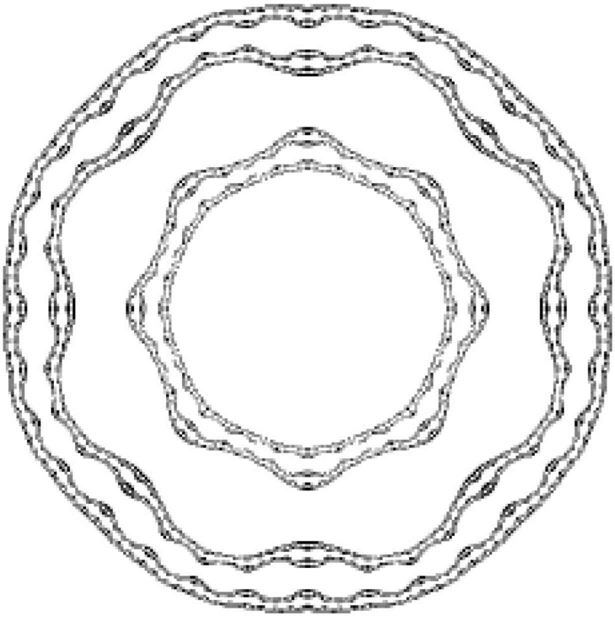


Figure 1: The Julia set for $z^4 + 0.04/z^4$ is a Cantor set of circles.

$\lambda = 0$ for which the Julia set is a Sierpinski curve. See Figure 2. A Sierpinski curve is an extremely rich topological space since this object is known to contain a homeomorphic copy of any one-dimensional, planar continuum, no matter how complicated this continuum is. It is also known that any two Sierpinski curves are homeomorphic [20]. However, from a dynamical systems point of view, it turns out that there are infinitely many dynamically distinct Sierpinski curve Julia sets in the sense that, if the parameters are drawn from disjoint open sets in the λ–plane, the corresponding maps are not topologically conjugate on their Julia sets.

When $n = 2$, $m = 2$ or $n = 2$, $m = 1$, there are many other interesting types of Julia sets in these families. For example, it is known [8] that there are infinitely many Julia sets in these families that have properties similar to a Sierpinski gasket. See Figure 3. These sets are topologically very different from the Sierpinski curves and it can be shown that, except for certain symmetric cases, these types of Julia sets are never homeomorphic.

In addition, in these two cases, there is a fundamental dichotomy for these rational maps that is similar in spirit to that for quadratic polynomials. This dichotomy states that if the critical points for these maps lie in the immediate basin of ∞, then the corresponding Julia set is a Cantor set, whereas if the critical points do not lie in this immediate basin, the Julia set is a connected set. The difference between the rational map case and the quadratic polynomial case is that the critical points for the rational maps may escape to ∞ without lying in the immediate basin of ∞, which is not possible for quadratic

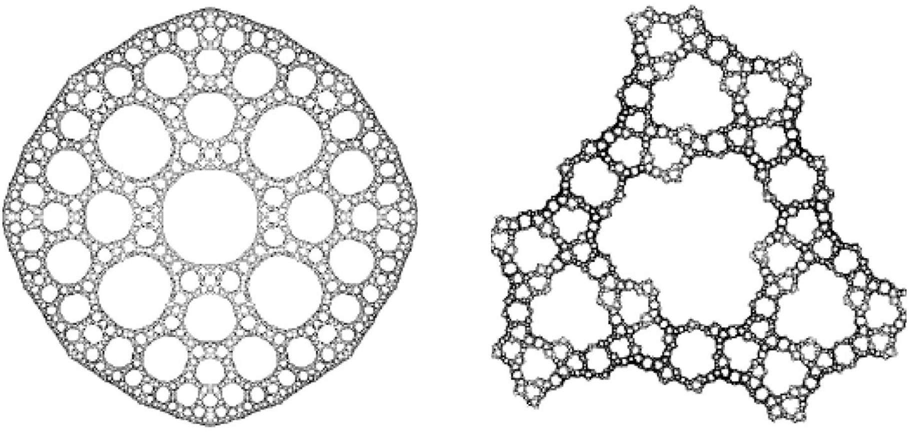


Figure 2: The Julia sets for (a) $z^2 - 0.06/z^2$, and (b) $z^2 + (-0.004 + 0.364i)/z$ are Sierpinski curves.

polynomials. As we show below, it is this situation that creates the Sierpinski curve Julia sets.

With this variety of different types of Julia sets in these families, it is little wonder that the parameter plane for these families is a rich topological object. Among other things, these parameter planes include infinitely many copies of "baby" Mandelbrot sets as well as other topologically interesting sets such as Cantor necklaces [3], [4]. See Figure 4.

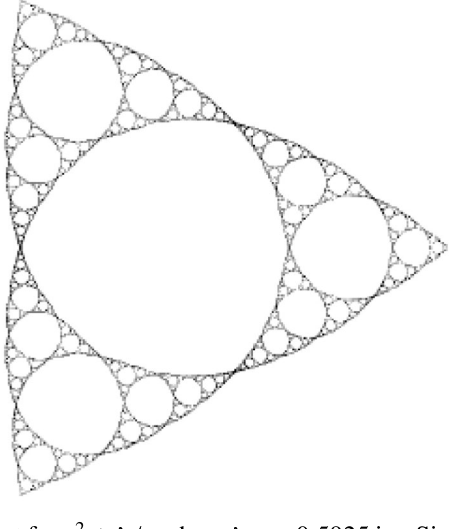


Figure 3: The Julia set for $z^2 + \lambda/z$ where $\lambda \approx -0.5925$ is a Sierpinski gasket.

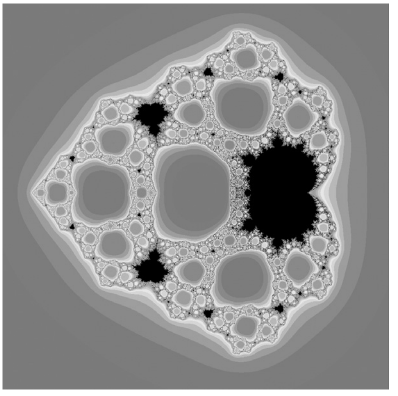


Figure 4: The parameter plane for $z^2 + \lambda/z^2$.

In this paper we restrict attention to the family of rational maps given by

$$F_\lambda(z) = z^2 + \frac{\lambda}{z^2},$$

although occasionally we will discuss the family

$$\tilde{F}_\lambda(z) = z^2 + \frac{\lambda}{z}.$$

Most of the results below hold for both families, though the proofs in the cases of F_λ and $\tilde{F}_\lambda$ are often quite different due to the presence of quite different symmetries in these two different families.

The authors wish to thank Pascale Roesch who made many fine suggestions concerning the original version of this paper.

Dedication. We are pleased to dedicate this paper to Bodil Branner, who is one of the finest mathematicians we have ever met. No, add to that: one of the finest **people** we have ever met.

1. Preliminaries. In this section we describe some of the basic properties of the family $F_\lambda(z) = z^2 + \lambda/z^2$. Observe that $F_\lambda(-z) = F_\lambda(z)$ and $F_\lambda(iz) = -F_\lambda(z)$ so that $F_\lambda^2(iz) = F_\lambda^2(z)$ for all $z \in \mathbb{C}$. Also note that 0 is the only pole for each function in this family. The points $(-\lambda)^{1/4}$ are prepoles for F_λ since they are mapped directly to 0. The four critical points for F_λ occur at $\lambda^{1/4}$. Note that $F_\lambda(\lambda^{1/4}) = \pm 2\lambda^{1/2}$ and $F_\lambda^2(\lambda^{1/4}) = 1/4 + 4\lambda$, so each of the four critical points lies on the same forward orbit after two iterations. We call the union of these orbits the *critical orbit* of F_λ.

Let $J = J(F_\lambda)$ denote the Julia set of F_λ. $J(F_\lambda)$ is the set of points at which the family of iterates of F_λ fails to be a normal family in the sense of Montel. Equivalently, $J(F_\lambda)$ is the closure of the set of repelling periodic points of F_λ (see [15]).

The point at ∞ is a superattracting fixed point for F_λ. Let B_λ be the immediate basin of attraction of ∞ and denote by β_λ the boundary of B_λ. The map F_λ has degree 2 at ∞ and so F_λ is conjugate to $z \mapsto z^2$ on B_λ if there is no critical point in B_λ. Otherwise, this conjugacy is defined only in a neighborhood of ∞. The basin B_λ is a (forward) invariant set for F_λ in the sense that, if $z \in B_\lambda$, then $F_\lambda^n(z) \in B_\lambda$ for all $n \geq 0$. The same is true for β_λ.

We denote by $K = K(F_\lambda)$ the set of points whose orbit under F_λ is bounded. In analogy with the situation for complex polynomials, we call K the filled Julia set of F_λ. K is given by $\mathbb{C} - \cup F_\lambda^{-n}(B_\lambda)$. Both J and K are completely invariant subsets in the sense that if $z \in J$ (resp. K), then $F_\lambda^n(z) \in J$ (resp. K) for all $n \in \mathbb{Z}$. The Julia set $J(F_\lambda)$ is the boundary of $K(F_\lambda)$; the proof is completely analogous to that for polynomials (see [15]).

Proposition (Four-fold Symmetry). *The sets B_λ, β_λ, $J(F_\lambda)$, and $K(F_\lambda)$ are all invariant under $z \mapsto iz$.*

Proof. We prove this for B_λ; the other cases are similar. Let $U = \{z \in B_\lambda \mid iz \in B_\lambda\}$. U is an open subset of B_λ. If $U \neq B_\lambda$, there exists $z_0 \in \partial U \cap B_\lambda$, where ∂U denotes the boundary of U. Hence $z_0 \in B_\lambda$ but $iz_0 \in \beta_\lambda$. It follows that $F_\lambda^n(iz_0) \in \beta_\lambda$ for all n. But since $F_\lambda^2(z_0) = F_\lambda^2(iz_0)$, it follows that $z_0 \notin B_\lambda$ as well. This contradiction establishes the result. □

There is a second symmetry present for this family. Consider the map $H_\lambda(z) = \sqrt{\lambda}/z$. Note that we have two such maps depending upon which square root of λ we choose. H_λ is an involution and we have $F_\lambda(H_\lambda(z)) = F_\lambda(z)$. As a consequence, H_λ preserves both J and K. The involution H_λ also preserves the circle S_λ of radius $|\lambda|^{1/4}$ and interchanges the interior and exterior of this circle. We call S_λ the *critical circle*. Note that S_λ contains all four critical points as well as the four prepoles, and each of the involutions H_λ fixes a pair of the critical points of F_λ that are located symmetrically about the origin.

Write $\lambda = \rho \exp(i\psi)$ and $z = \rho^{1/4} \exp(i\theta) \in S_\lambda$. Then we compute

$$\begin{aligned} F_\lambda(z) &= \rho^{1/2}(\exp(2i\theta) + \exp(i(\psi - 2\theta))) \\ &= \rho^{1/2}((\cos(2\theta) + \cos(\psi - 2\theta)) + i(\sin(2\theta) + \sin(\psi - 2\theta))). \end{aligned}$$

If we set $x = \cos(2\theta) + \cos(\psi - 2\theta)$ and $y = \sin(2\theta) + \sin(\psi - 2\theta)$, then a computation shows that

$$\frac{d}{d\theta}\left(\frac{y}{x}\right) = 0.$$

Hence the image of the critical circle under F_λ is a line segment passing through the origin. F_λ maps S_λ onto this line in four-to-one fashion, except at the two endpoints, which are the critical values $\pm 2\sqrt{\lambda}$.

Note also that H_λ interchanges the circles centered at the origin and having radii $|\lambda|^{1/4}r$ and $|\lambda|^{1/4}/r$. Moreover, F_λ maps each of these two circles onto an ellipse that surrounds the image of the critical circle.

2. The Fundamental Dichotomy. We briefly recall the situation for the family of quadratic polynomials $Q_c(z) = z^2 + c$. Each map Q_c has a single critical point at 0 and so, like F_λ, Q_c has a single critical orbit. The fate of this orbit leads to the well known fundamental dichotomy for quadratic polynomials:

1. If $Q_c^n(0) \to \infty$, then $J(Q_c)$ is a Cantor set;
2. but if $Q_c^n(0) \not\to \infty$, then $J(Q_c)$ is a connected set.

The set of parameter values for which the quadratic Julia sets are connected is the well known Mandelbrot set. Our goal in this section is to prove a similar result in the case of F_λ. We remark that there is a more general form of this result called the escape trichotomy that holds in the more genral case of maps of the form $z^n + \lambda/z^m$. We refer to [6] for details.

Before stating this result, note that, unlike the quadratic case, there are two distinct ways that the critical orbit of F_λ may tend to ∞. One possibility is that one (and hence all) of the critical points lie in the immediate basin B_λ. The second possibility is that the critical points do not lie in B_λ but eventually map into B_λ. For quadratic polynomials this second possibility does not occur.

Theorem.

1. *If one and hence all of the critical points of F_λ lie in B_λ, then $J(F_\lambda)$ is a Cantor set.*
2. *If the finite critical points of F_λ do not lie in B_λ, then both $J(F_\lambda)$ and $K(F_\lambda)$ are compact and connected. In particular, if the finite critical points do not lie in B_λ but are mapped to B_λ by F_λ^n for some $n \geq 1$, then $J(F_\lambda)$ and $K(F_\lambda)$ are compact, connected, and locally connected sets.*

Proof. The proof that $J(F_\lambda)$ is a Cantor set when all critical points lie in B_λ is standard. See, for example, [15]. So suppose that no finite critical point lies in B_λ. Then we may extend the conjugacy between F_λ and z^2 to all of B_λ and so B_λ is a simply connected open set in $\overline{\mathbb{C}}$ and we have $F_\lambda \mid B_\lambda$ is two-to-one.

Since 0 is a pole of order two, there is an open, simply connected set T_λ containing 0 and having the property that F_λ maps T_λ in two-to-one fashion onto B_λ. This follows since each of the two involutions H_λ interchange B_λ and T_λ. One checks easily that T_λ possesses four-fold symmetry. Note that B_λ and T_λ are necessarily disjoint open sets. Note also that none of the critical points reside in T_λ. This follows since, if $\lambda^{1/4} \in T_\lambda$, then $-(\lambda^{1/4}) \in T_\lambda$ as well, and hence F_λ would be four-to-one on T_λ.

It is also true that none of the critical values lie in T_λ. We will assume this fact for now and provide a proof in the next section.

Now let $K_0 = \overline{\mathbb{C}} - B_\lambda$. K_0 is compact and connected with boundary β_λ. Let $K_1 = K_0 - F_\lambda^{-1}(B_\lambda) = K_0 - T_\lambda$. Since B_λ and T_λ are disjoint, K_1 is compact and connected. Now consider $F_\lambda^{-1}(T_\lambda)$. By our assumption above, none of the critical points of F_λ lies in $F_\lambda^{-1}(T_\lambda)$. Hence each component of $F_\lambda^{-1}(T_\lambda)$ is mapped in one-to-one fashion onto T_λ. Therefore, there are four disjoint components in this set, and each component is open, simply connected, and disjoint from both T_λ and B_λ.

We remark here that, if the critical points were to lie in $F_\lambda^{-1}(T_\lambda)$, then $F_\lambda^{-1}(T_\lambda)$ would be an annulus, not a collection of disks. This is the situation we will rule out later.

Thus we have that $K_2 = K_1 - F_\lambda^{-1}(T_\lambda)$ is a compact, connected set. Now we continue removing preimages of T_λ. Let $K_3 = K_2 - F_\lambda^{-2}(T_\lambda)$. If the orbit of the critical points of F_λ do not escape to ∞, then each component of $F_\lambda^{-2}(T_\lambda)$ is mapped one-to-one onto a component of $F_\lambda^{-1}(T_\lambda)$ and so $F_\lambda^{-2}(T_\lambda)$ consists of 16 simply connected open sets, each of which is disjoint from the previously removed open sets. Hence K_3 is compact and connected. Continuing in this fashion, assuming that the critical points do not escape, the components of $F_\lambda^{-n}(T_\lambda)$ ($n \geq 2$) are mapped one-to-one onto components of $F_\lambda^{-n+1}(T_\lambda)$ and so $F_\lambda^{-n}(T_\lambda)$ consists of 4^{n-1} simply connected open sets, each of which is disjoint from the previously removed open sets. Hence, inductively, $K_n = K_{n-1} - F_\lambda^{-n+1}(T_\lambda)$ is compact and connected for all n. Therefore $K(F_\lambda) = \cap K_n$ is compact and connected. Since J is the boundary of K, J is also compact and connected.

If, on the other hand, one of the critical points lies in $F_\lambda^{-2}(T_\lambda)$, we claim that all of the preimages of T_λ under F_λ^2 are still open, simply connected, and disjoint, and that four of them are mapped two-to-one onto their images while the rest are mapped in one-to-one fashion.

To see this, suppose that one of the critical points, say c_λ, lies in a component V of $F_\lambda^{-2}(T_\lambda)$ that is mapped by F_λ onto a component of $F_\lambda^{-1}(T_\lambda)$. Call the image component W. Then a second critical point, $-c_\lambda$, is also mapped into W. Consequently, the set $-V$ containing $-c_\lambda$ is also mapped onto W. Now either V and $-V$ are disjoint, simply connected, and mapped two-to-one onto W, or else they form the same component of the preimage of W. In the latter case, there can be no other critical points in this component, for $\pm ic_\lambda$ are mapped to $-W$, which is disjoint from W. Hence F_λ is a degree four mapping onto a disk with exactly two critical points. This cannot happen by the Riemann-Hurwitz formula. Therefore the former case holds, and $\pm c_\lambda$ lie in disjoint components of $F_\lambda^{-2}(T_\lambda)$. Similarly, $\pm ic_\lambda$ lie in disjoint components of this set.

Thus, $F_\lambda^{-2}(T_\lambda)$ consists of a collection of non-intersecting, simply connected open sets lying in K_2. Hence K_3 is compact and connected. We now continue in the same fashion to show inductively that K_n is compact and connected. Therefore $K(F_\lambda) = \cap K_n$ is compact and connected, as is its boundary, $J(F_\lambda)$. This shows that J and K are compact and connected if the critical orbit escapes to ∞ but the critical points do not lie in B_λ. Also, since no critical points accumulate on J, the map is hyperbolic and so it is known [15] that J is locally connected.

We emphasize once again that the critical points for F_λ may not lie in B_λ yet the critical orbits may eventually enter B_λ. As shown in the above proof, this implies that the critical orbit passes through T_λ, the disjoint preimage of B_λ that contains the origin. We call T_λ the *trap door*, since any orbit that enters T_λ immediately "falls through" it and enters the basin at ∞. In this case we have a connected Julia set. In fact, we shall show in the next section that $J(F_\lambda)$ is a Sierpinski curve in the special case where this occurs and $|\lambda|$ is sufficiently small.

We denote the set of parameter values for which $J(F_\lambda)$ is connected by $\mathcal{M}$; $\mathcal{M}$ is called the *connectedness locus* for this family.

Proposition. *The connectedness locus lies on or inside the circle of radius* $3/16+\sqrt{2}/8 \approx 0.364$ *centered at* 0 *in the parameter plane.*

Proof. The critical values are given by $\pm 2\sqrt{\lambda}$. Consider the circle of radius $2|\sqrt{\lambda}|$ centered at 0. If z lies on this circle, we have

$$|F_\lambda(z)| \geq 4|\lambda| - \frac{1}{4}.$$

Note that

$$4|\lambda| - \frac{1}{4} > 2|\sqrt{\lambda}|$$

provided that

$$16|\lambda|^2 - 6|\lambda| + \frac{1}{16} > 0,$$

and this occurs if $|\lambda| > 3/16+\sqrt{2}/8$. Hence F_λ maps the circle of radius $2|\sqrt{\lambda}|$ strictly outside itself for these λ-values.

Now the involution H_λ takes this circle to the circle of radius $1/2$ centered at the origin, and we have $2|\sqrt{\lambda}| > 1/2$ since $|\lambda| > 3/16+\sqrt{2}/8$. It follows that F_λ maps the exterior of $|z| = 2|\sqrt{\lambda}|$ into itself in two-to-one fashion, so it follows that this entire region must lie in B_λ. Hence the critical values lie in B_λ in this case. From the proof of the fundamental dichotomy, the critical points cannot lie in the trap door and so they too must reside in B_λ. Therefore λ does not belong to $\mathcal{M}$. $\square$

Note that these estimates for the size of $\mathcal{M}$ are the best possible, for if $\lambda = \lambda^* = -3/16 - \sqrt{2}/8$, then we have

$$\begin{aligned} F_\lambda^2(c_\lambda) &= -\frac{1+\sqrt{2}}{2} \\ F_\lambda^3(c_\lambda) &= \frac{1+\sqrt{2}}{2} \\ &= F_\lambda^4(c_\lambda). \end{aligned}$$

Hence the critical orbit lands on a fixed point for this particular λ-value. In $\mathcal{M}$, λ^* lies at the leftmost tip of the connectedness locus along the negative real axis. We shall deal with this particular λ-value and other related values in Section 4.

In analogy with the situation for quadratic functions, we also have the following *escape criterion*, though one can give significantly better estimates for this using the previous result.

Proposition. *Suppose* $|\lambda| \leq 2$ *and* $|z| \geq 2$*. Then* $|F_\lambda^n(z)| > (1.5)^n|z|$*, and therefore* $z \in B_\lambda$*.*

Proof. We have

$$|F_\lambda(z)| \geq |z|^2 - \frac{|\lambda|}{|z|^2} \geq |z|^2 - \frac{1}{2} \geq 1.5|z|.$$

Inductively, we have

$$|F_\lambda^n(z)| > (1.5)^n|z|$$

and the result follows.

3. Sierpinski Curve Julia Sets. In this section we describe the case where the critical points of F_λ have orbits that tend to ∞, but the critical points themselves do not lie in the immediate basin of ∞. The main result here is:

Theorem. *Suppose the critical orbit of* F_λ *tends to* ∞*, but the critical points do not lie in the immediate basin of* ∞*. Then* $J(F_\lambda)$ *is a Sierpinski curve.*

A *Sierpinski curve* is an interesting topological space that is, by definition, homeomorphic to the well known Sierpinski carpet fractal [12]. The Sierpinski carpet is a set that is obtained by starting with a square in the plane and dividing it into nine congruent subsquares, each of which has sides of length $1/3$ the size of the original square. Then the open middle square is removed, leaving eight subsquares in the original square. Then this process is repeated: remove the open middle third from each remaining square. This leaves 64 subsquares, each of which is $1/9$ the size of the original. Continuing, in the limit, the space that is obtained is the Sierpinski carpet. See [3].

It is straightforward to show that the Sierpinski carpet is a compact, connected, locally connected, nowhere dense subset of the plane. Moreover, each of the complementary domains (the removed open squares) is bounded by a simple closed curve that is disjoint from the boundary of every other complementary domain. It is known [20] that these properties characterize a Sierpinski curve: any planar set that is compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by pairwise disjoint simple closed curves is homeomorphic to the Sierpinski carpet. Hence any two Sierpinski curve Julia sets drawn from the family F_λ are homeomorphic. The interesting topology arises from the fact that a Sierpinski curve contains a homeomorphic copy of any one-dimensional plane continuum [17].

As an illustration of the proof of the theorem, we provide the details in the special case where $|\lambda| < 3^3/4^4 \approx 0.1$. For a proof for arbitrary λ, we refer to [7].

Proposition. *Suppose that $|\lambda| < 3^3/4^4$. Then the boundary of B_λ is a simple closed curve.*

Proof. Suppose z lies on the circle of radius 3/4 centered at the origin. Then

$$|F_\lambda(z)| \leq |z|^2 + \frac{|\lambda|}{|z|^2} < \frac{9}{16} + \frac{3}{16} = 3/4$$

since $|\lambda| < 3^3/4^4$. Hence F_λ maps the circle of radius 3/4 in two-to-one fashion onto an ellipse lying inside this circle. Also note that all critical points of F_λ lie inside this circle.

Let A_λ denote the annular region between the circle of radius 3/4 and its preimage that lies outside this circle. Note that F_λ has degree two on A_λ as well as in the entire exterior region $r \geq 3/4$ since all critical points lie in $r < 3/4$. Let U_λ denote the disk in the complement of A_λ that contains the origin.

We now use quasiconformal surgery to modify F_λ to a new map E_λ which agrees with F_λ in the region outside A_λ but which is conjugate to $z \mapsto z^2$ in the interior of U_λ with a fixed point at the origin. To obtain E_λ, we first replace F_λ in U_λ with the map $z \mapsto z^2$ on $|z| < 3/4$. Then we extend E_λ to A_λ so that the new map is continuous and

1. maps A_λ two-to-one onto $U_\lambda - E_\lambda(U_\lambda)$;
2. agrees with E_λ on the inner boundary of A_λ;
3. and agrees with F_λ on the outer boundary of A_λ.

The map E_λ is continuous and has degree 2 with two superattracting fixed points, one at 0 and one at ∞. We define a new complex structure on $\mathbb{C}$ that is preserved by E_λ in the usual manner. Hence E_λ is quasiconformally conjugate to z^2 on all of $\mathbb{C}$. Therefore the boundary of the basin of attraction of ∞ for E_λ is a simple closed curve. Since E_λ agrees with F_λ in the exterior portion of A_λ containing ∞, the same is true for F_λ. This proves that β_λ is a simple closed curve when $|\lambda| < 3^3/4^4$. □

In particular, when $|\lambda| < 3^3/4^4$, since all of the critical points lie inside the circle of radius 3/4 centered at the origin, the only way the critical orbits can escape to ∞ in this case is by passing through the trap door. Therefore we have:

Corollary. *The region $|\lambda| < 3^3/4^4$ lies in the interior of the connectedness locus.*

Before moving on, we use the above technique to fill in the hole we left in the previous section:

Proposition. *If the finite critical points are not in B_λ (so $B_\lambda \neq T_\lambda$), then the critical values of F_λ do not lie in T_λ.*

Proof. Let $\pm v_\lambda$ denote the critical values of F_λ. Recall that $F_\lambda(v_\lambda) = F_\lambda(-v_\lambda)$. Suppose for the sake of contradiction that the critical values of F_λ lie in the trap door. Let γ be a simple closed curve in B_λ that separates both ∞ and $F_\lambda(v_\lambda)$ from the boundary of B_λ. Let Γ be the closed disk in the Riemann sphere that is bounded by γ and contains both ∞ and $F_\lambda(v_\lambda)$. Let Λ denote the preimage of Γ in T_λ. Λ contains 0 and $\pm v_\lambda$ in its interior.

Consider $F_\lambda^{-1}(\Lambda)$. We claim that $F_\lambda^{-1}(\Lambda)$ is an annulus that is disjoint from T_λ and also surrounds T_λ. We first observe that $F_\lambda^{-1}(\Lambda)$ must be a connected set. If this were not the case, then this set would consist of at most two components, since each preimage of Λ necessarily contains at least two of the four critical points. So suppose $F_\lambda^{-1}(\Lambda)$ consists of two disjoint components, C_+ and C_-. If the critical point c_λ belongs to C_+, then $-c_\lambda$ belongs to C_- since both of these points are mapped to the same critical value. Then the critical point ic_λ belongs to one of these sets, say C_+, so $-ic_\lambda \in C_-$.

Now apply the involution H_λ to C_+. Recall that there are two such involutions, and each fixes a pair of critical points. We choose the one that fixes c_λ and $-c_\lambda$. Since $F_\lambda(H_\lambda(z)) = F_\lambda(z)$, we have $H_\lambda(C_+) = C_+$ and $H_\lambda(C_-) = C_-$. But $H_\lambda(ic_\lambda) = -ic_\lambda$ and so we cannot have $H_\lambda(C_+) = C_+$. This contradiction shows that $F_\lambda^{-1}(\Lambda)$ cannot consist of two disjoint components.

So let $C = F_\lambda^{-1}(\Lambda)$. Since C contains 4 critical points and is mapped with degree 4 onto a simply connected region, the Riemann-Hurwitz formula implies that C must be an annulus. As in the previous Proposition, we may replace F_λ by a new map that agrees with F_λ outside C and is globally conjugate to $z \mapsto z^2$. As before, this proves that the boundary of B_λ is a simple closed curve. So too is its preimage, the boundary of T_λ.

Now the region between B_λ and T_λ is an annulus A that is bounded by these two simple closed curves. Let Q denote the preimage of T_λ lying in A. As above, Q is an annulus. A is then the union of three subannuli, A_{in}, Q, and A_{out}, where A_{in} is the inner annulus between T_λ and Q, and A_{out} is the outer annulus between Q and B_λ. F_λ maps both A_{in} and A_{out} two-to-one onto A. Therefore the modulus of A_{in} and A_{out} is one-half the modulus of A. But the modulus of the third annulus Q is positive, and the modulus of A is the sum of the moduli of A_{in}, A_{out}, and Q. This yields a contradiction. Hence the critical values cannot lie in T_λ as claimed. □

We now use this result to prove:

Theorem. *Suppose $|\lambda| < 3^3/4^4$ and that the critical points of F_λ tend to ∞ but do not lie in the the immediate basin B_λ of ∞. Then $J(F_\lambda) = K(F_\lambda)$ is a Sierpinski curve.*

Proof. It suffices to show that $J(F_\lambda)$ is compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by simple closed curves that are disjoint. The fact that both J and K are compact, connected, and locally connected was shown in the previous section. Since all of the critical orbits tend to ∞, it follows that $J = K$ and hence, using standard properties of the Julia set, J is nowhere dense.

It therefore suffices to show that all of the complementary domains are bounded by disjoint simple closed curves. By the earlier Proposition, B_λ is bounded by a simple closed curve β_λ lying strictly outside the circle of radius $3/4$. Using the involution H_λ, the boundary of the trap door is given by $H_\lambda(\beta_\lambda)$, and so this region is bounded by a simple closed curve disjoint from β_λ.

As in the previous section, the preimage of T_λ consists of four simply connected open sets whose boundaries are simple closed curves that are mapped onto the boundary of T_λ, which we denote by τ_λ. The boundaries of these components are disjoint from β_λ, since this curve is invariant under F_λ. They are disjoint from τ_λ since the boundary of the trap door is mapped to β_λ whereas the boundary of the components are mapped to τ_λ, and we know that $\tau_\lambda \cap \beta_\lambda = \emptyset$. Finally, the boundary of each component is disjoint from any other such boundary for a point in the intersection would necessarily be a critical point. If this were the case, then the critical orbit would eventually map to β_λ, contradicting our assumption that the critical orbit tends to ∞. Hence the first preimages of T_λ are all bounded by simple closed curves that are disjoint from each other as well as the boundaries of B_λ and T_λ. Continuing in this fashion, we see that the preimages $F_\lambda^{-n}(T_\lambda)$ are similarly bounded by simple closed curves that are disjoint from all earlier preimages of β_λ. This gives the result. □

While these Sierpinski curve Julia sets are all homeomorphic, it is known that there are infinitely many open sets of parameter values $\mathcal{O}_j$ having the property that, if λ_1 and λ_2 belong to distinct $\mathcal{O}_j$'s, then F_{λ_1} and F_{λ_2} are not topologically conjugate on their respective Julia sets. See [2]. The basic reason for this lack of topological conjugacy is the fact that, in different $\mathcal{O}_j$'s, the number of iterations for the critical orbit to enter B_λ is different.

4. Sierpinski Gasket Julia sets. In this section we turn our attention to a different type of Julia set that occurs for certain members of the family F_λ. We assume in this section that the critical points of F_λ all lie on the boundary of the immediate basin of ∞ and that the critical orbit is preperiodic. We call such maps *Misiurewicz-Sierpinski* maps, or MS maps, for short.

Since all of the critical points are preperiodic, the Julia set of an MS map is the complement of the union of all preimages of B_λ, just as in the Sierpinski curve case. Hence we may construct this set inductively as in the proof of the fundamental dichotomy. Let K_0 denote $\mathbb{C} - B_\lambda$. It is known that the boundary of K_0 is a simple closed curve (for a proof, see [6]). Let $K_1 = K_0 - T_\lambda$ and for $k \geq 1$ set

$$K_{k+1} = K_k - F_\lambda^{-k}(T_\lambda).$$

Then

$$J(F_\lambda) = \bigcap_{k=0}^{\infty} K_k.$$

This construction yields a very different type of Julia set in the case of MS maps. To see this, note first that, using the involution H_λ, the critical points lie in the boundary τ_λ of the trap door as well as in β_λ. It can be shown [6] that, in fact, the critical points are the only points lying in the intersection of β_λ and τ_λ. Thus, when we remove the trap door from K_0 to form K_1, we are essentially removing an open *generalized square*, a region bounded by a simple closed curve with four *corners* that are the four critical points. The four corners divide the boundary of the square into four curves that we call *edges*. In particular, if we remove the four critical points from K_1, then the resulting set consists of four disjoint sets $I'_0, \ldots, I'_3$. We assume that I'_0 contains the fixed point p_λ that lies in β_λ and that the other I'_j are indexed in the counterclockwise direction. Let I_j denote the closure of I'_j, so that I_j is just I'_j with two critical points added. Then, by four-fold symmetry, F_λ maps I_j in one-to-one fashion onto K_0.

Since there are no critical points in any of the preimages of the trap door, K_{k+1} is obtained by removing 4^k generalized squares from K_k. Each of these removed squares is mapped homeomorphically onto the trap door by F_λ^k and hence each has exactly four corners lying in the boundary of K_k. By definition, these corners are the preimages of the critical points.

This process is reminiscent of the deterministic process used to construct the Sierpinski gasket (sometimes called the Sierpinski triangle). To construct this set, we start with a triangle and remove a middle triangle so that three congruent triangles remain, each of which meets the other two triangles at a unique point. We then continue this process, removing 3^k triangles at the k^{th} stage. In the limit we obtain the Sierpinski gasket. In analogy with this construction, and despite the fact that the removed sets are generalized squares rather than triangles we call the Julia set of an MS map a *generalized Sierpinski gasket.*

If we consider the degree three family

$$\tilde{F}_\lambda(z) = z^2 + \frac{\lambda}{z},$$

then there are analogous MS parameters for which $J(\tilde{F}_\lambda)$ is a generalized Sierpinski gasket where "triangles" are removed instead of squares. For example, when $\lambda \approx -0.5925$, the Julia set of $\tilde{F}_\lambda$ is actually homeomorphic to the Sierpinski triangle. See Figure 3.

We have the following result. See [8] for the complete proof.

Theorem. *Suppose F_λ and F_μ are two MS maps with $\lambda \neq \mu$ and the imaginary parts of both λ and μ are positive. Then $J(F_\lambda)$ is not homeomorphic to $J(F_\lambda)$.*

We make the assumption in this theorem that the imaginary parts of the parameters are positive because the Julia sets of F_λ and $F_{\overline{\lambda}}$ are easily seen to be homeomorphic. The Julia sets of two MS maps in the family F_λ are displayed in Figure 5. In the first example, the parameter value $\nu = -3/16 - \sqrt{2}/8 \approx -0.36428$ lies at the leftmost tip of the connectedness locus. The critical points can be clearly identified as the four corners of the trap door and are mapped after three iterations onto the repelling fixed point p_ν that lies in β_ν. The second example corresponds to $\mu \approx -0.01965 + 0.2754i$

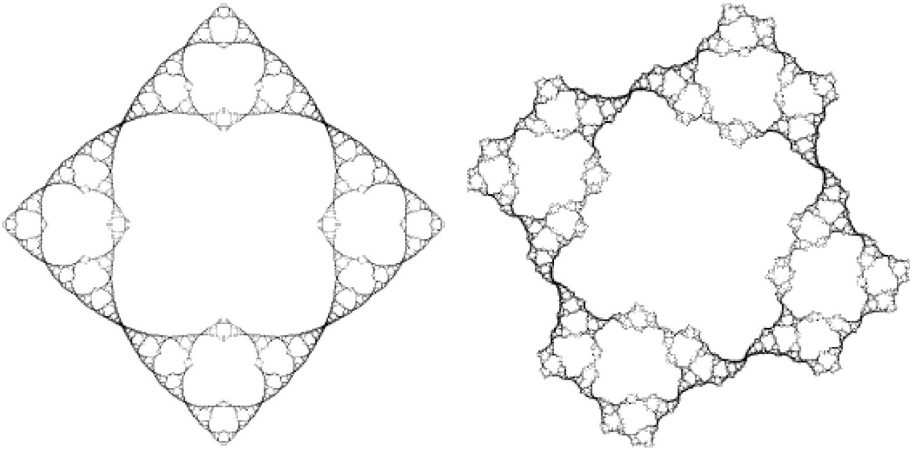


Figure 5: $J(F_\lambda)$ for $\lambda \approx -0.36428$ and $\lambda \approx -0.01965 + 0.2754i$.

for which the critical points are mapped to p_μ after four iterations. Rather than present the full details of the proof of the above theorem, we will illustrate the principal ideas using these two examples.

Note first that, in both of these images, every preimage of the boundary of the trap door seems to have two corners lying in the boundary of a previous preimage. This configuration holds true for every MS map as we show next.

Proposition. Let τ_λ^k be the union of all of the components of $F_\lambda^{-k}(\tau_\lambda)$ and let A be a particular component in τ_λ^k with $k \geq 1$. Then exactly two of the corner points of A lie in a particular edge of a single component of τ_λ^{k-1}.

Proof. The case $k = 1$ is seen as follows. Recall that $J(F_\lambda)$ is contained in the union of four closed sets $I_0, \ldots, I_3$ that meet only at the critical points and that are mapped by F_λ in one-to-one fashion onto $\mathbb{C} - B_\lambda$. Hence F_λ maps each $I_j \cap J(F_\lambda)$ for $j = 0, \ldots, 3$ in one-to-one fashion onto all of $J(F_\lambda)$, with $F_\lambda(I_j \cap \beta_\lambda)$ mapped onto one of the two halves of β_λ lying between two critical values (which, by assumption, are not equal to any of the critical points). Hence $F_\lambda(I_j \cap \beta_\lambda)$ contains exactly two critical points. Similarly, $F_\lambda(I_j \cap \tau_\lambda)$ maps onto the other half of β_λ and so also meets two critical points. The preimages of these four critical points are precisely the corners of the component of τ_λ^1 that lies in I_j. Thus we see that each component of τ_λ^1 meets the boundary of one of the I_j's in two points lying in β_λ and two points lying in τ_λ. In particular, two of the corners lie in the edge of τ_λ that meets I_j.

Now consider a component in τ_λ^k with $k > 1$. F_λ^k maps each component in τ_λ^k onto τ_λ and therefore F_λ^{k-1} maps the components in τ_λ^k onto one of the four components of τ_λ^1. Since each of these four components meets a particular edge of τ_λ in exactly two

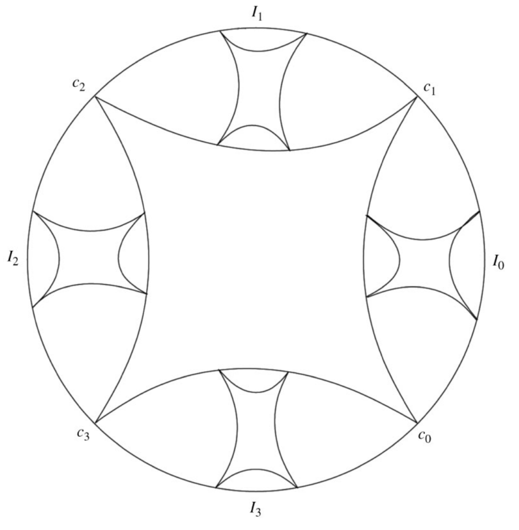


Figure 6: A topological representation of the boundaries β_λ, τ_λ, the four components of τ_λ^1 and the critical points. These curves satisfy the above configuration for every MS map.

corner points, it follows that each component of τ_λ^k meets an edge of one of the components of τ_λ^{k-1} in exactly two corner points as claimed.

Figure 6 provides a caricature of β_λ, τ_λ and τ_λ^1 which is valid for any MS λ-value. We seek a topological criterion that allows us to conclude that the Julia sets of two MS maps are not homeomorphic. The following result provides a topological characterization of the critical points that is helpful in this regard (see [8] for the proof).

Proposition. *Suppose F_λ is an MS map. The four corners of the trap door is the only set of four points in the Julia set whose removal disconnects $J(F_\lambda)$ into exactly four components. Any other set of four points removed from $J(F_\lambda)$ will yield at most three components.*

Suppose now that λ and μ are both MS parameters. If there exists a homeomorphism $h : J(F_\lambda) \to J(F_\mu)$, then it follows from the Proposition that:

1. h maps the corners of τ_λ to the corners of τ_μ, and
2. the corners of each component of $F_\lambda^{-k}(\tau_\lambda)$ are mapped to the corners of a unique component of $F_\mu^{-k}(\tau_\mu)$.

As we will show below, the configuration of the corners with respect to the curve β_λ provides enough information to determine when two Julia sets are homeomorphic. This configuration, on the other hand, is completely determined by the orbit of the critical points.

To specify such an orbit, we define the itinerary $S(z)$ of a point $z \in J(F_\lambda)$ in the natural way by recording how its orbit meanders through the regions $I_0, \ldots, I_3$. That is, $S(z) = (s_0 s_1 s_2 \ldots)$ where each s_k is an integer j between 0 and 3 that specifies which I_j the point $F_\lambda^k(z)$ lies in. So that itineraries are unique, we modify the I_j slightly by removing one of the critical points from each I_j (so this set is no longer closed). In particular, let c_0 be the critical point of F_λ that lies in the fourth quadrant and in I_0. Then let $c_1 = ic_0$, $c_2 = -c_0$, and $c_3 = -ic_0$. We specify that only the critical point c_j now lies in I_j. Using these I_j's, the fixed point p_λ lies in I_0 so its itinerary is given by $S(p_\lambda) = \overline{0}$. Its preimage $q_\lambda \in \beta_\lambda$ lies in I_2; hence $S(q_\lambda) = 2\overline{0}$. Since the critical point c_j lies in I_j only, we have a well determined itinerary for each c_j. For example if $\lambda = \nu \approx -0.36428$, then one computes easily that the itinerary of the critical point c_1 is given by $S(c_1) = 112\overline{0}$. Using the symmetries of the map F_λ, it is easy to see that $S(c_0) = 032\overline{0}$, $S(c_2) = 232\overline{0}$ and $S(c_3) = 312\overline{0}$. When $\lambda = \mu \approx -0.01965 + 0.2754i$, $S(c_1) = 1112\overline{0}$ and thus $S(c_2) = 2312\overline{0}$, $S(c_3) = 3112\overline{0}$ and $S(c_0) = 0312\overline{0}$.

For each MS λ–value we define the *k–skeleton* of the Julia set, denoted by $\mathbb{J}(F_\lambda, k)$, as the union of β_λ, τ_λ and τ_λ^j for $j = 1, \ldots, k$. The k–skeleton not only provides the configuration of the first k preimages of τ_λ along β_λ, but if we define $\mathbb{J} = \lim_{k \to \infty} \mathbb{J}(F_\lambda, k)$, then the closure of $\mathbb{J}$ is equal to the Julia set, $J(F_\lambda)$.

We may construct a homeomorphism $\varphi = \varphi_\lambda$ that maps β_λ to the unit circle S^1 and "straightens" any other curve in $\mathbb{J}(F_\lambda, k)$ to a smooth curve, except at the images of the corners. Let

$$M(F_\lambda, k) = S^1 \cup \varphi(\tau_\lambda) \cup \varphi(\tau_\lambda^1) \cup \ldots \cup \varphi(\tau_\lambda^k).$$

The set $M(F_\lambda, k)$ represents a topological model in the plane of the k–skeleton of the Julia set. Since F_λ acts like $z \to z^2$ when restricted to β_λ, the model inherits the dynamics of the angle doubling map $D(\theta) = 2\theta \bmod 2\pi$ in S^1. Thus, to any point $z \in \beta_\lambda$ we can naturally associate an angle $\theta(z) \in [0, 2\pi]$ given by the angle of $\varphi_\lambda(z)$ in S^1. We may assume that $M(F_\lambda, k)$ satisfies the same symmetry relations as $J(F_\lambda)$ and that the four half-open regions I_j are mapped to corresponding regions in $M(F_\lambda, k)$.

To illustrate the construction of the model, consider our first example $\nu \approx -0.36428$. In this case, the critical point c_1 has itinerary $(112\overline{0})$. Hence $\theta(c_1) = \pi/4$ and, by symmetry, $\theta(c_2) = 3\pi/4$, $\theta(c_3) = 5\pi/4$ and $\theta(c_0) = 7\pi/4$. Since every model inherits the configuration of the Julia set for MS maps, each component of $\varphi(\tau_\nu^1)$ has two corners lying on an edge of $\varphi(\tau_\nu)$ and the two remaining corners must lie on $S^1 = \varphi(\beta_\nu)$. We let $x_0 = \varphi(F_\nu^{-1}(c_0))$ and $x_1 = \varphi(F_\nu^{-1}(c_1))$, so that x_0 and x_1 are the two corners that lie in $I_0 \cap S^1$. Similarly, we let x_2 and x_3 be the corresponding corners in $I_0 \cap \varphi(\tau_\nu)$. So we have $\theta(x_0) = D^{-1}(7\pi/4) = 7\pi/8$ and $\theta(x_1) = D^{-1}(\pi/4) = \pi/8$. A rotation by a multiple of $\pi/2$ provides the angles of the corners lying on $I_j \cap S^1$. For example, if w_0 and w_1 are the corresponding corners of $\varphi(\tau_\nu^1)$ lying in $I_1 \cap S^1$, then $\theta(w_0) = 3\pi/8$ and $\theta(w_1) = 5\pi/8$. Figure A shows in fact the angles along S^1 in $M(F_\nu, 1)$.

To construct $M(F_\nu, 2)$, we will determine first the configuration of the corners of a single component in $\varphi(\tau_\nu^2) \subset I_0$. Let U be the component of τ_ν^2 that is contained in

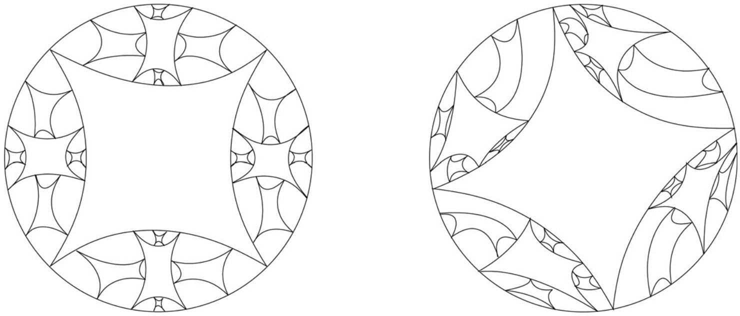


Figure 7: The model $M(F_\nu, 2)$ for $\nu \approx -0.36428$ is displayed to the left and $M(F_\mu, 2)$ with $\mu \approx -0.01965 + 0.2754i$ is displayed to the right. Note that $\theta(c_1(\nu)) = \pi/4$ and $\theta(c_1(\mu)) = \pi/16$.

the "triangle" T_ν defined by x_1, $\varphi(c_1)$ and x_2. Label the corners of U by y_0, y_1, y_2 and y_3 and assume y_2 and y_3 lie in the edge $[x_2, x_1] \subset \tau_\nu^1$. To compute the location of the remaining two corners, we first note the arc $[x_1, \varphi(c_1)] \subset S^1$ is mapped under D to an arc γ in $I_1 \cap S^1$. Clearly $\theta(\gamma) = [\pi/4, \pi/2]$ and thus, the corner w_0 lies in γ. Pulling back γ into I_0 by D we obtain the corner $y_0 \in [x_1, \varphi(c_1)]$.

A similar argument can be applied to the arc $\alpha = [\varphi(c_1), x_2]$ to obtain the location of y_1. Since D is not defined in this arc, consider first $\varphi^{-1}(\alpha) = [c_1, F_\nu^{-1}(c_2)]$ in $I_0 \cap J(F_\nu)$. This arc is mapped by F_ν to $[F_\nu(c_1), c_2] \subset I_1 \cap \beta_\nu$. Then, the homeomorphism φ sends $F_\nu(\varphi^{-1}(\alpha))$ onto $[D(\varphi(c_1)), \varphi(c_2)] \subset I_1 \cap S^1$. This arc has angles $[\pi/2, 3\pi/4]$ and thus, the corner w_1 lies in it. Pulling back this point by the proper composition of maps yields the point $y_1 \in \alpha$.

A similar process can be carried out to obtain $M(F_\mu, 2)$ for our second example where $\mu \approx -0.01965 + 0.2754i$. See [8] for the details. Figure 7 shows the models for the two MS maps discussed above. Both models display the configuration of the corner points with respect to the angles of corners along S^1.

To show the Julia sets of F_ν and F_μ are not homeomorphic, we proceed by contradiction. Assume there exists a homeomorphism $h : J(F_\nu) \to J(F_\mu)$. Recall that a homeomorphism must send critical points to critical points and corners of components in τ_ν^k to corners of components in τ_μ^k. Without loss of generality, assume that h maps $c_1(\nu)$ to $c_1(\mu)$ and $I_0(\nu)$ onto $I_0(\mu)$.

Restricting h to the 2-skeletons of the Julia sets, we can define a new homeomorphism $\widetilde{h}$ defined by the following diagram

$$\begin{array}{ccc} \mathbb{J}(F_\nu, 2) & \xrightarrow{\varphi_\nu} & M(F_\nu, 2) \\ h\big\downarrow & & \big\downarrow\widetilde{h} \\ \mathbb{J}(F_\mu, 2) & \xrightarrow{\varphi_\nu} & M(F_\mu, 2). \end{array}$$

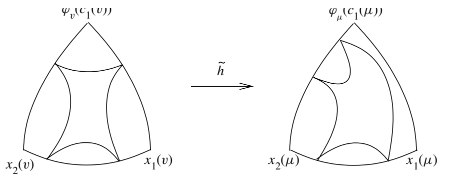


Figure 8: The map $\widetilde{h}$ must preserve the configurations of the corners along the edges of T_ν and T_μ.

Let T_ν denote the triangle in $M(\nu, 2)$ with vertices $x_1(\nu)$, $\varphi_\nu(c_1(\nu))$ and $x_2(\nu)$. This triangle contains a unique component $U(\nu) \subset \tau_\nu^1$ that defines a configuration of its corners along the edges of T_ν. Define T_μ analogously (see Figure 8). Since there exists an edge in T_μ that contains no corners of $U(\mu)$, $\widetilde{h}$ cannot possibly send the configuration given by $U(\nu)$ to the configuration of $U(\mu)$, and we have reached a contradiction.

The procedure to prove that any two MS maps have topologically distinct Julia sets is similar in spirit to this construction, although in the general case one needs to proceed to the k-skeleton to show this where k may be large.

5. The Connectedness Locus. Recall that the connectedness locus $\mathcal{M}$ for the family F_λ is the set of all λ-values for which $J(F_\lambda)$ is connected. In this section, we mimic the Douady-Hubbard proof [11] to show that $\overline{\mathbb{C}} - \mathcal{M}$ is conformally equivalent to the open unit disk $\mathbb{D}$. See also [18].

As we have already noted, the point at infinity is a superattracting fixed point. Consequently, the map F_λ is locally, analytically conjugate to the map $z \mapsto z^2$ in a neighborhood of ∞. In our case, there exists an analytic homeomorphism ϕ_λ defined in a neighborhood of ∞ such that

1. $\phi_\lambda(\infty) = \infty$
2. $\phi'_\lambda(\infty) = 1$
3. $\phi_\lambda \circ F_\lambda(z) = (\phi_\lambda(z))^2$.

This homeomorphism is often called the Böttcher coordinate of F_λ. For $|z| \geq \max\{|\lambda|, 2\}$, one can use the triangle inequality to show that $|F_\lambda(z)| > \frac{3}{2}|z|$, and in this case ϕ_λ is given by the infinite product representation

$$\phi_\lambda(z) = z \prod_{k=0}^{\infty} \left(1 + \frac{\lambda}{{z_k}^4}\right)^{1/2^{k+1}}$$

where $z_k = F_\lambda^k(z)$.

Let $O^{-(0)}$ denote the backwards orbit of the pole. That is,

$$O^-(0) = \bigcup_{k=0}^{\infty} F_\lambda^{-k}(0).$$

Associated to ϕ_λ is the nonnegative rate-of-escape function

$$G_\lambda : \mathbb{C} - O^-(0) \to \mathbb{R}$$

given by

$$G_\lambda(z) = \lim_{k\to\infty} \frac{1}{2^k} \log_+ |z_k|,$$

where $\log_+$ is the maximum of log and 0. This function has the following properties:

1. Let A_λ be the entire basin of ∞. Then $G_\lambda^{-1}(0) = \mathbb{C} - A_\lambda$.
2. $G_\lambda(z_{k+1}) = 2G_\lambda(z_k)$.
3. $G_\lambda(iz) = G_\lambda(z)$.
4. $G_\lambda(H_\lambda(z)) = G_\lambda(z)$.
5. G_λ is harmonic on $A_\lambda - O^{-(0)}$.
6. $G_\lambda(z) = \log|z| + \text{bounded terms as } |z| \to \infty$.
7. $G_\lambda(z) = \log|\phi_\lambda(z)|$ if $\phi_\lambda(z)$ is defined.

The nonzero level sets of G_λ are called the *equipotential curves* for A_λ.

We know that the immediate basin B_λ of ∞ assumes one of two topological types:

1. If the critical points are not in B_λ, then B_λ is simply connected, and the domain of ϕ_λ can be extended to all of B_λ. Therefore, $F_\lambda : B_\lambda \to B_\lambda$ is a two-to-one branched cover with the branch point at ∞.
2. If the finite critical points are in B_λ, then the Julia set of F_λ is a Cantor set, and B_λ is the Fatou set.

In case 1, the involution H_λ determines the remaining inverse image T_λ of B_λ. That is, if $T_\lambda = H_\lambda(B_\lambda)$, then $F_\lambda : T_\lambda \to B_\lambda$ is also a two-to-one branched cover of B_λ with branch point at 0, and

$$F_\lambda^{-1}(B_\lambda) = B_\lambda \cup T_\lambda.$$

The distinction above results in a nice division of parameter space for the family $F_\lambda(z)$ into two disjoint subsets. We now focus on those values of λ for which case 2 holds, i.e., those λ for which the Julia set is topologically conjugate to the one-sided shift on four symbols. Consequently, we call this subset $\mathbb{C} - \mathcal{M}$ of parameter space the shift locus. Given $\lambda \in \mathbb{C} - \mathcal{M}$, let L_λ denote the component of $\{z \mid G_\lambda(z) > G_\lambda(\lambda^{1/4})\}$ that contains ∞. Then ϕ_λ extends naturally to all of L_λ. We mimic the Douady-Hubbard uniformization of the complement of the Mandelbrot set by defining the map $\Phi : \mathbb{C} - \mathcal{M} \to \mathbb{C} - \overline{\mathbb{D}}$ as

$$\Phi(\lambda) = \phi_\lambda\left(\frac{1}{4} + 4\lambda\right).$$

There are three things that need to be verified to show that Φ determines a conformal equivalence between $\mathbb{C} - \mathcal{M}$ and $\mathbb{C} - \overline{\mathbb{D}}$:

1. The map Φ is holomorphic.
2. It extends to a holomorphic map from $\mathbb{C}-\mathcal{M}\cup\{\infty\}$ to $\overline{\mathbb{C}}-\overline{\mathbb{D}}$.
3. The extension is a proper map of degree one.

First, Φ is holomorphic because $\frac{1}{4}+4\lambda$ lies in L_λ for all $\lambda\in\mathbb{C}-\mathcal{M}$ and $\phi_\lambda(z)$ varies analytically in both z and λ.

For step 2, we use the infinite product representation of ϕ_λ. For $|\lambda|>\frac{9}{16}$, we have

$$\phi_\lambda\left(\frac{1}{4}+4\lambda\right)=\left(\frac{1}{4}+4\lambda\right)\prod_{k=0}^{\infty}\left(1+\frac{\lambda}{\lambda_k^4}\right)^{1/2^{k+1}}$$

where $\lambda_k=F_\lambda^k(\frac{1}{4}+4\lambda)$. From this expression, we see that

$$\frac{\Phi(\lambda)}{\lambda}\to 4$$

as $\lambda\to\infty$, and we can extend Φ to a holomorphic map from $\mathbb{C}-\mathcal{M}\cup\{\infty\}$ to $\overline{\mathbb{C}}-\overline{\mathbb{D}}$ by setting $\Phi(\infty)=\infty$.

The map Φ is proper if $|\Phi(\lambda)|\to 1$ as $\lambda\to\partial\mathcal{M}$. To show this, we compute $G_\lambda(\frac{1}{4}+4\lambda)$ as $\lambda\to\partial\mathcal{M}$ using two lemmas.

Lemma. *The boundary $\partial\mathcal{M}$ of the connectedness locus is contained in the annulus*

$$\frac{3^3}{4^4}\le|\lambda|\le\frac{3+2\sqrt{2}}{16}.$$

The first inequality was proved in Section 3 and the second in Section 2.

Lemma. *Let $\lambda_k=F_\lambda^k(\frac{1}{4}+4\lambda)$ for $k=0,1,2,\ldots$. Then $\partial\mathcal{M}$ is a subset of*

$$\{\lambda\,|\,|\lambda_k|\le 2 \text{ for all } k=0,1,2,\ldots\}.$$

Proof. From the results of Section 2 we know that if $|\lambda|>2$, then $\lambda\in\mathbb{C}-\mathcal{M}$, so we may assume that $|\lambda|\le 2$.

Suppose $|\lambda_k|>2$ for some k. Then the escape criterion from Section 2 guarantees that $\lambda_k\in B_\lambda$. Either $\lambda\in\mathbb{C}-\mathcal{M}$, or $\lambda\in\mathcal{M}$ with $\lambda_j\in T_\lambda$. In the first case we have $\lambda\notin\partial\mathcal{M}$. In the latter case, an open neighborhood of λ also has $\lambda_j\in T_\lambda$ and therefore λ is in the interior of $\mathcal{M}$. □

For each $\lambda\in\mathbb{C}-\mathcal{M}$, let m_λ correspond to the last iterate such that $|\lambda_{m_\lambda}|\le 2$. Note that the previous lemma implies that $m_\lambda\to\infty$ as $\lambda\to\partial\mathcal{M}$. For a fixed $\lambda\in\mathbb{C}-\mathcal{M}$, we drop the subscript on m_λ, and we estimate $G_\lambda(\frac{1}{4}+4\lambda)$ by considering

$$\sqrt[2^n]{|\lambda_m+n|}$$

for $n=1,2,\ldots$. Note that

$$\sqrt[2^n]{|\lambda_{m+n}|}=\sqrt[2^n]{\frac{|\lambda_{m+n}|}{|\lambda_{m+n-1}|^2}}\,\sqrt[2^n-1]{\frac{|\lambda_{m+n-1}|}{|\lambda_{m+n-2}|^2}}\,\cdots\,\sqrt{\frac{|\lambda_{m+1}|}{|\lambda_m|^2}}|\lambda_m|.$$

Assuming $|\lambda| < 1$, we estimate all but the last two factors by

$$\sqrt[2^k]{\frac{|\lambda_{m+k}|}{|\lambda_{m+k-1}|^2}} \leq \sqrt[2^k]{1+\frac{1}{|\lambda_{m+k-1}|^4}}.$$

If $k > 1$, we know that $|\lambda_{m+k-1}| > 2$, and we apply Bernoulli's inequality $\sqrt[k]{1+x} \leq 1 + x/k$ to obtain

$$\sqrt[2^k]{\frac{|\lambda_{m+k}|}{|\lambda_{m+k-1}|^2}} \left(1+\frac{1}{16\cdot 2^k}\right).$$

The term

$$\sqrt{\frac{|\lambda_{m+1}|}{|\lambda_m|^2}}$$

requires special attention because $|\lambda_m| \leq 2$. For this part of the argument, it is convenient to work within the disk $|\lambda + \frac{1}{16}| \leq \frac{1}{2}$. Fix a λ within this disk. Let A_0 be the annulus bounded by the circle of radius 2 centered at the origin and the ellipse that is its image under F_λ.

Since the critical values of F_λ cannot be in A_0, A_0 has two preimages. One preimage A_1 has the circle of radius 2 centered at the origin as one boundary component, and the other preimage of A_0 is $H(A_1)$. Note that A_1 lies outside the unit disk.

Similarly, the critical values of F_λ cannot be in A_1, and therefore A_1 has two preimages. One preimage A_2 has a boundary in common with A_1 and the other preimage is $H(A_2)$.

Proceeding in this manner, we can produce annuli A_n such that

1. F_λ maps A_n onto A_{n-1} in a two-to-one fashion, and
2. one boundary component of A_n is also a boundary component of A_{n-1} as long as the critical values of F_λ do not lie in A_{n-1}.

If the critical values do not lie in A_n for some n, then the Julia set of F_λ is connected and

$$\{z \,|\, |z| \geq 2\} \cup \bigcup_{n=1}^{\infty} A_n$$

exhaust the immediate basin of infinity. Consequently, $\lambda_m \in A_1$ and $\lambda_{m+1} \in A_0$ for $\lambda \in \mathbb{C} - \mathcal{M}$. Therefore, we have

$$\sqrt{\frac{|\lambda_{m+1}|}{|\lambda_{m2}|}} \leq \frac{\sqrt{265}}{8}.$$

Combining these two estimates with the fact that $|\lambda_m| < 2$, we obtain

$$\sqrt[2^n]{|\lambda_{m+n}|} \leq \frac{\sqrt{265}}{4} \prod_{k=2}^{n} \left(1+\frac{1}{16\cdot 2^k}\right) < \frac{\sqrt{265}}{4} \prod_{k=2}^{\infty} \left(1+\frac{1}{16\cdot 2^k}\right).$$

Since the right-hand side of this inequality converges to some number C (independent of those λ within the annulus under consideration), we have

$$\lim_{n\to\infty} \sqrt[2^n]{|\lambda_{m+n}|} \leq C.$$

This inequality implies that $G_\lambda(\lambda_m) \leq \log C$. Consequently,

$$2^m G_\lambda\left(\frac{1}{4}+4\lambda\right) \leq \log C.$$

Since $m \to \infty$ as $\lambda \to \partial\mathcal{M}$, we conclude that

$$G_\lambda\left(\frac{1}{4}+4\lambda\right) \to 0$$

as $\lambda \to \partial\mathcal{M}$, and Φ is a proper map.

We see that Φ has degree one by noting that $\Phi^{-1}(\infty) = \{\infty\}$. We have proved:

Theorem. *The complement $\overline{\mathbb{C}} - \mathcal{M}$ of the connectedness locus is conformally equivalent to a disk in the Riemann sphere.*

References

[1] P. BLANCHARD, Complex Analytic Dynamics on the Riemann Sphere, *Bull. Amer. Math. Soc.* Vol. II, No. 1, 1984, 85–141.

[2] P. BLANCHARD, R. L. DEVANEY, D. M. LOOK, P. SEAL and Y. SHAPIRO, Sierpinski Curve Julia Sets and Singular Perturbations of Complex Polynomials. *Ergodic Theory and Dynamical Systems* **25** (2005).

[3] R. L. DEVANEY, Cantor and Sierpinski, Julia and Fatou, Complex Topology Meets Complex Dynamics. *Notices Amer. Math. Soc.* **51** (2004), 9–15.

[4] R. L. DEVANEY, Baby Mandelbrot Sets Adorned with Halos in Families of Rational Maps. *To appear in Complex Dynamics: Twenty Five Years After the Appearance of the Mandelbrot set*. Amer.~.Math.~Soc.

[5] R. L. DEVANEY, K. JOSIC and Y. SHAPIRO, Singular Perturbations of Quadratic Maps. *International Journal of Bifurcations and Chaos.* **14** (2004), 161–169

[6] R. L. DEVANEY, D. M. LOOK and D UMINSKY, The Escape Trichotomy for Singularly Perturbed Rational Maps. To appear in Indiana University Mathematics Journal.

[7] R. L. DEVANEY and D. M. LOOK, A Criterion for Sierpinski Curve Julia Sets for Rational Maps. To appear in *Topology Proceedings.*

[8] R. L. DEVANEY, M. MORENO ROCHA and S. SIEGMUND, Rational Maps with Generalized Sierpinski Gasket Julia Sets. Preprint, 2003.

[9] A. DOUADY and J. HUBBARD. Itération des Polynômes quadratiques complexes, *C.R. Acad. Sci. Paris*, t. 29, Serie I–1982, pp. 123–126.

[10] A. DOUADY and J. HUBBARD, On the Dynamics of Polynomial-like Mappings, *Ann. Scient., Éc. Norm. Sup.* 4*e* *séries*, t. **18** (1985), 287–343.

[11] A. DOUADY and HUBBARD, J. Etude Dynamique des Polynômes Complexes. *Publ. Math. D'Orsay* (1984).

[12] K. KURATWOSKI, *Topology*. Vol. **2**. Academic Press, New York, 1968.

[13] C. McMullen, Automorphisms of Rational Maps. *Holomorphic Functions and Moduli*. Vol. **1**. Math. Sci. Res. Inst. Publ. 10. Springer, New York, 1988.
[14] C. McMullen, The Classification of Conformal Dynamical Systems. *Current Developments in Mathematics*. Internat. Press, Cambridge, MA, 1995, 323–360.
[15] J. Milnor, Dynamics in One Complex Variable. Vieweg, 1999.
[16] J. Milnor and Tan Lei. A "Sierpinski Carpet" as Julia Set. Appendix F in Geometry and Dynamics of Quadratic Rational Maps. *Experiment. Math.* **2** (1993), 37–83.
[17] S. Nadler, *Continuum Theory: An Introduction*. Marcel Dekker, New York, 1992.
[18] N. Steinmetz, *Rational Iteration.* W. de Gruyter, Berlin, 1993.
[19] D. Sullivan, Quasiconformal Maps and Dynamical Systems I, Solutions of the Fatou-Julia Problem on Wandering Domains. *Ann. Math.* **122** (1985), 401–418.
[20] G. T. Whyburn, Topological Characterization of the Sierpinski Curve. *Fund. Math.* **45** (1958), 320–324.
[21] B. Wittner, On the Bifurcation Loci of Rational Maps of Degree Two. Thesis, Cornell University (1988).

Dept. of Mathematics
Boston University
111 Cummington St.
Boston, MA 02215 USA
e-mail: paul@bu.edu
e-mail: bob@bu.edu

On capture zones for the family $f_\lambda(z) = z^2 + \lambda/z^2$

P. Roesch

Introduction. This paper originates from the Bodilfest and it is a pleasure to dedicate it to Bodil Branner. It is motivated by the picture[1] shown in Figure 1 of the parameter plane of the family $f_\lambda(z) = z^2 + \lambda/z^2$. At the Bodilfest, Devaney presented this particular parameter plane describing the various behaviours that can appear in the family (see [B-D et al.]). Besides the unbounded component, all the hyperbolic components look like holes, we call them *capture zones*, or seem to belong to copies of the Mandelbrot set. We solve in this article the conjecture of Devaney that there are exactly 4^{n-1} capture zones of order n. Moreover we prove that there is no copy of the Mandelbrot set attached at the cusp to a capture zone. The ideas of the proof are not new (see [F, McM, M2]). We construct a parametrization of the capture zones using surgery. This construction can be adapted to various families. With this parametrization one sees that there are as many capture zones of order n as there are solutions of a certain polynomial equation (describing the center of the components).

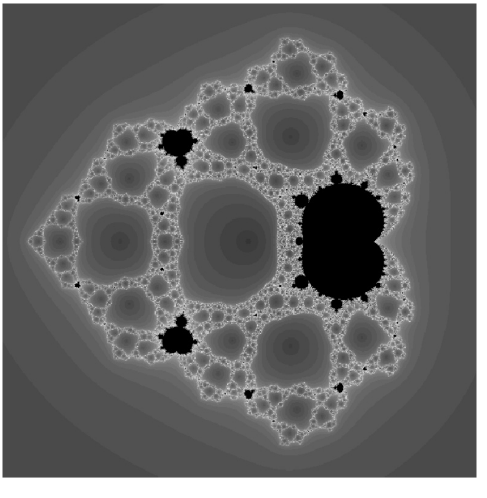


Figure 1: Parameter plane of the family f_λ.

[1] The pictures here are made with a very nice program of Dan Sorensen who I would like to thank.

1. The family and its properties after [B-D et al.]. For $\lambda \neq 0$, the family $f_\lambda(z) = z^2 + \lambda/z^2$ is formed by the rational maps of degree 4 that have the following properties:

- ∞ is a super-attracting fixed point (in particular a critical point);
- the only pole 0 is a critical point;
- the other critical points are $\lambda^{1/4}$ and they have the same forward orbit: $f_\lambda(\lambda^{1/4}) = (2\lambda)^{1/2}$ and $f_\lambda^2(\lambda^{1/4}) = \frac{1}{4} + 4\lambda$;
- f_λ has symmetries: $f_\lambda(iz) = -f_\lambda(z)$ and $f_\lambda(-z) = f_\lambda(z)$.

Definition 1.1. Let $A_\lambda = \{z \mid f_\lambda^n(z) \underset{n\to\infty}{\longrightarrow} \infty\}$ be the basin of ∞, B_λ the immediate basin of ∞, i.e., the connected component of A_λ containing ∞, and let T_λ denote the preimage of B_λ containing 0. We consider the sets $\mathcal{H}_0 = \{\lambda \mid \frac{1}{4} + 4\lambda \in B_\lambda\}$ and

$$\mathcal{H}_n = \left\{\lambda \mid f_\lambda^n\left(\frac{1}{4} + 4\lambda\right) \in B_\lambda \text{ and } f_\lambda^{n-1}\left(\frac{1}{4} + 4\lambda\right) \notin B_\lambda\right\} \text{ if } n > 0.$$

Since $\cup_{n\neq 0}\mathcal{H}_n$ is the set of parameters for which the critical points eventually fall into B_λ, the connected components are called capture zones.

Throughout this section we refer for the proofs to the article [B-D et al.].

Theorem 1.2 [B-D et al.]. *For $\lambda \in \mathcal{H}_0$, the critical points all lie in B_λ and the Julia set is a Cantor set.*

Lemma 1.3 [B-D et al.]. *For $\lambda \notin \mathcal{H}_0$ the following properties hold:*

- *B_λ contains a unique critical point which is ∞;*
- *the preimage $f_\lambda^{-1}(B_\lambda)\backslash B_\lambda$ is connected—so $f_\lambda^{-1}(B_\lambda)\backslash B_\lambda = T_\lambda$;*
- *T_λ is a topological disk, and 0 is the only critical point in T_λ;*
- *there are no critical values in T_λ.*

The first three properties of this lemma follow from topological arguments, the last one is an analytic argument.

Theorem 1.4 [D-L]. *For $\lambda \in \mathcal{H}_n$ with $n > 0$, the boundaries of B_λ and its preimages are Jordan curves.*

For $\lambda \in \mathcal{H}_n$, $n > 0$, the map f_λ is hyperbolic and $J(f_\lambda)$ is locally connected.

Let φ_λ be the Böttcher coordinate near infinity i.e., the unique holomorphic germ such that $\varphi_\lambda \circ f_\lambda(z) = (\varphi_\lambda(z))^2$, $\varphi_\lambda(\infty) = \infty$ and $\varphi_\lambda(z) \sim_\infty z$.

Remark 1.5. For $\lambda \notin \mathcal{H}_0$, the conjugacy φ_λ extends continuously to all B_λ and is holomorphic in (λ, z) over $\{(\lambda, z) \mid z \in B_\lambda, \lambda \notin \overline{\mathcal{H}_0}\}$ (see [B]).

Theorem 1.6 [B-D et al.]. *The map* $\Phi : \mathcal{H}_0 \to \mathbf{C}\backslash\overline{\mathbf{D}}$ *defined by* $\Phi(\lambda) = \varphi_\lambda(\frac{1}{4} + 4\lambda)$ *is a conformal homeomorphism.*

2. Parametrization of the capture zones.

2.1. The Definition. The classical idea to parametrize the capture zones is to use the Böttcher coordinate of the first forward iterate of the critical point that enters the immediate basin of attraction. However, here the iterates of the critical points before landing on the basin B_λ pass through T_λ which is a ramified double cover of B_λ. So we choose to take the position of the orbit of the critical points in T_λ using a "lifted" Böttcher coordinate.

Lemma 2.1. *For* $\lambda \notin \mathcal{H}_0$*, there exists a holomorphic map* $\widetilde{\varphi}_\lambda : T_\lambda \to \mathbf{D}$ *satisfying*

$$\varphi_\lambda \circ f_\lambda = \frac{1}{(\widetilde{\varphi}_\lambda)^2}$$

where $\varphi_\lambda : B_\lambda \to \overline{\mathbf{C}}\backslash\overline{\mathbf{D}}$ *is the Böttcher coordinate near* ∞*. Two such maps differ by multiplication by* -1.

Proof. The coverings $f_\lambda : T_\lambda\backslash\{0\} \to B_\lambda\backslash\{\infty\}$ and $z \mapsto 1/z^2$ from $\mathbf{D}\backslash\{0\}$ to $\mathbf{C}\backslash\overline{\mathbf{D}}$ have the same action on the fundamental group. Hence fixing a point (x, y) with $y \neq 0$ and x in the fiber of $\varphi_\lambda^{-1}(f_\lambda(y))$ (i.e., $1/x^2 = \varphi_\lambda^{-1}(f_\lambda(y))$), there exists a unique lift $\widetilde{\varphi}_\lambda : T_\lambda\backslash\{0\} \to \mathbf{D}\backslash\{0\}$ of $\varphi_\lambda : B_\lambda\backslash\{\infty\} \to \mathbf{C}\backslash\overline{\mathbf{D}}$ with $\widetilde{\varphi}_\lambda(x) = y$. Finally, $\widetilde{\varphi}_\lambda$ extends to 0 by continuity. □

Lemma 2.2. *No connected component of* $\mathcal{H}_n$ *with* $n \neq 0$ *separates* 0 *from* ∞.

Proof. Let $\mathcal{U}$ be a connected component of $\mathcal{H}_n$. We prove that if $\mathcal{U}$ intersects $\mathbf{R}^+$ then $n = 0$. Let λ be a parameter in $\mathbf{R}^+ \cap \mathcal{U}$. The map $f_\lambda : \mathbf{R}^+ \to \mathbf{R}^+$ is real. There is only one critical point on $\mathbf{R}^+$ denoted by c_λ and the graph on $\mathbf{R}^+$ is like a parabola. If the parabola stays above the diagonal, the forward orbit of every point tends to ∞, so $\mathbf{R}^+$ is a path in B_λ connecting c_λ to ∞ and therefore $\lambda \in \mathcal{H}_0$. If the parabola crosses the diagonal, there are two fixed points. The largest, β, is repelling and $[\beta, +\infty[$ is a path in B_λ (as before). Let $0 < \beta' < c_\lambda$ be a preimage of β, then $]0, \beta']$ is a preimage of $[\beta, +\infty[$. If the critical value $f_\lambda(c_\lambda)$ is smaller than β' then its image $f_\lambda^2(c_\lambda) = 1/4 + 4\lambda$ is in $[\beta, +\infty[\subset B_\lambda$ and $\lambda \in \mathcal{H}_0$ by Lemma 1.3 (last item). If $f_\lambda(c_\lambda)$ is between β' and β, the interval $[f_\lambda(c_\lambda), \beta]$ is stable by f_λ since $f_\lambda(c_\lambda)$ is the minimum and $f_\lambda(\beta) = f_\lambda(\beta') = \beta$. This case is impossible since the critical point cannot escape to infinity ($\lambda \notin \mathcal{H}_n$). □

Lemma 2.3. *Let* $\mathcal{U}$ *be a connected component of* $\mathcal{H}_n$ *with* $n \neq 0$ *and let* $\widetilde{\varphi}_\lambda$ *be the lift of* φ_λ *with* $\widetilde{\varphi}'_\lambda(0) = 1/\sqrt{\lambda}$ *(see Lemma* 2.1*), where* $\sqrt{\lambda}$ *is a branch of the square root defined on* $\mathcal{U}$*. Then the map* $(\lambda, z) \mapsto \widetilde{\varphi}_\lambda(z)$ *is holomorphic where defined (i.e., for* $\lambda \in \mathcal{U}$ *and* $z \in T_\lambda$*).*

Proof. The map $\widetilde{\varphi}_\lambda$ is locally defined as $\widetilde{\varphi}_\lambda(z) = 1/(\sqrt{\varphi_\lambda(f_\lambda(z))})$ where $\sqrt{\ }$ is the chosen branch. Near a point $(\lambda_0, z_0) \in \mathcal{U} \times T_{\lambda_0} \setminus \{0\}$ the function $(\lambda, z) \mapsto \varphi_\lambda(z)$ is holomorphic so $\widetilde{\varphi}_\lambda(z)$ is locally defined as a composition of holomorphic maps. Also notice that if $H_\lambda(z) = \sqrt{\lambda}/z$, then $\widetilde{\varphi}_\lambda(z) = 1/(\varphi_\lambda \circ H_\lambda(z))$. □

Definition 2.4. Let $\mathcal{U}$ be a connected component in $\mathcal{H}_n$ with $n \neq 0$. Define the parametrization $\Phi : \mathcal{U} \to \mathbf{D}$ by $\Phi(\lambda) = \widetilde{\varphi}_\lambda(f_\lambda^{n-1}(1/4 + 4\lambda))$.

Remark 2.5. For $\lambda \in \mathcal{U}$, the point $f_\lambda^{n-1}(1/4+4\lambda)$ belongs to T_λ, so $\widetilde{\varphi}_\lambda(f_\lambda^{n-1}(1/4+4\lambda))$ is well defined and holomorphic in λ.

2.2. Isomorphism via surgery.

Proposition 2.6. *The* map $\Phi : \mathcal{U} \to \mathbf{D}$ *defined by* $\Phi(\lambda) = \widetilde{\varphi}_\lambda(f_\lambda^{n-1}(1/4 + 4\lambda))$ *is a conformal isomorphism.*

Proof. The map Φ is holomorphic as it is a composition of holomorphic maps. In Lemma 2.8 below we show that Φ is proper, so it is a ramified covering. In fact, Φ has no ramification points since, by Lemma 2.9 below we have that Φ admits locally continuous sections. Therefore the covering $\Phi : \mathcal{U} \to \mathbf{D}$ is a homeomorphism. □

Lemma 2.7 [B-D et al.]. *For* $|\lambda| \leq 2$ *the immediate basin* B_λ *contains* $\mathbf{C} \setminus 2\mathbf{D}$ *and, if* $|\lambda| \geq 2$*, then* $\lambda \in \mathcal{H}_0$.

Lemma 2.8. *The map* $\Phi : \mathcal{U} \to \mathbf{D}$ *is proper for any connected component* $\mathcal{U}$ *of* $\mathcal{H}_n$*,* $n \neq 0$.

Proof. The proof is similar to the one for the Mandelbrot set (see [D-H]).

Let $\lambda_k \in \mathcal{U}$ be a sequence of parameters tending to $\lambda_0 \in \partial\mathcal{U}$. Our aim is to prove that $|\Phi(\lambda_k)| = 1/|\varphi_{\lambda_k}(f_{\lambda_k}^n(1/4 + 4\lambda_k))|^2$ tends to 1.

In what follows, we restrict attention to $\mathcal{F} = (\mathbf{C} \setminus \mathcal{H}_0)$. We define $G_\lambda(z) = \log|\varphi_\lambda(z)|$ if $z \in B_\lambda$ and $G_\lambda(z) = 0$ else. We will prove that $W_\varepsilon = \{(\lambda, z) \in \mathcal{F} \times \mathbf{C} \mid G_\lambda(z) < \epsilon\}$ is a neighborhood of (λ_0, z_0) in $\mathcal{F} \times \mathbf{C}$, so that it contains (λ_k, z_k) for k large, where the points $z_k = f_{\lambda_k}^n(1/4 + 4\lambda_k)$ tend to $z_0 = f_{\lambda_0}^n(1/4 + 4\lambda_0)$.

The complement $(\mathcal{F} \times \mathbf{C}) \setminus W_\varepsilon$ is exactly $\{(\lambda, z) \in \mathcal{F} \times \mathbf{C} \mid \log|\varphi_\lambda(z)| \geq \epsilon\}$. By Lemma 2.7, the map $\Psi(\lambda, z) = (\lambda, \varphi_\lambda(z))$ sends $\mathcal{F} \times (\mathbf{C} \setminus 2\overline{\mathbf{D}})$ into $\mathcal{F} \times (\mathbf{C} \setminus \overline{\mathbf{D}})$ and is injective. Moreover its restriction to $\mathcal{F} \times (\mathbf{C} \setminus 4\overline{\mathbf{D}})$ is continuous. Indeed, $\breve{\varphi}_{(z)} = (1/\breve{\varphi}_\lambda(4/z))$ (the map in the chart at infinity) satisfies $|(\breve{\varphi}_\lambda)'(z)| \leq 1$ for $z \in \overline{\mathbf{D}}$ (by the Cauchy formula) so that $(\breve{\varphi}_\lambda)$ is an equicontinuous family for $\lambda \in \mathcal{F}$. By the uniqueness of the conjugacy at ∞ there is a unique accumulation point for a sequence φ_{λ_n} with $\lambda_n \to \lambda_0$. Hence the convergence is uniform on every compact subset of $\mathbf{C} \setminus 4\overline{\mathbf{D}}$ and this implies the continuity of Ψ.

The image $\Psi(\mathcal{F} \times (\mathbf{C} \backslash 4\mathbf{D}))$ contains a tube $\mathcal{F} \times (\mathbf{C} \backslash R\mathbf{D})$ for some $R > 0$. Indeed, for $\lambda \in \mathcal{F}$, there exists r_λ such that $\breve{\varphi}_\lambda(\overline{\mathbf{D}}) \supset \overline{r_\lambda \mathbf{D}}$. The existence of R follows from the equicontinuity of $\breve{\varphi}_\lambda$ and the compactness of $\mathcal{F}$. So $\Psi^{-1}(\mathcal{F} \times (\mathbf{C} \backslash R\mathbf{D}))$ is a closed subset of $\mathcal{F} \times (\mathbf{C} \backslash 2\mathbf{D})$.

Because $\{(\lambda, z) \in \mathcal{F} \times \mathbf{C} \mid \log|\varphi_\lambda(z)| \geq \epsilon\} = \{(\lambda, z) \in \mathcal{F} \times \mathbf{C} \mid \log|\varphi_\lambda(f_\lambda^p(z))| \geq 2^p \epsilon\}$, the complement $\mathcal{F} \times \mathbf{C} \backslash W_\epsilon$ is $\{(\lambda, z) \in \mathcal{F} \times \mathbf{C} \mid f_\lambda^p(z) \in B_\lambda \text{and} F^p(\lambda, z) \in \Psi^{-1}(\mathbf{C} \backslash e^{2^p \epsilon}\mathbf{D})\}$ with $F(\lambda, z) = (\lambda, f_\lambda(z))$ for $(\lambda, z) \in \mathbf{C} \times \mathbf{C}$. Thus $(\mathcal{F} \times \mathbf{C}) \backslash W_\epsilon$ is exactly the connected component that intersects $\{-1/16\} \times [2, +\infty[$ of $F^{-p}(\Psi^{-1}(\mathbf{C} \backslash e^{2^p \epsilon}\mathbf{D}))$, which is closed in $\mathcal{F} \times \mathbf{C}$ for $e^{2^p \epsilon} > R$ (since F is clearly continuous). □

Lemma 2.9. *The map $\Phi : \mathcal{U} \to \mathbf{D}$ admits locally continuous sections, for any connected component $\mathcal{U}$ of $\mathcal{H}_n$ with $n \neq 0$.*

Proof. Let λ_0 be a parameter in $\mathcal{U}$ and $\zeta_0 = \Phi(\lambda_0)$. We construct, for $\epsilon > 0$ small enough, a continuous function $\lambda(\zeta)$ satisfying $\Phi(\lambda(\zeta)) = \zeta$ for $|\zeta - \zeta_0| < \epsilon$ as follows: We deform f_{λ_0} into a map g_ζ so that the orbit of the critical points has a Böttcher coordinate ζ (instead of ζ_0). The deformation is taken to be continuous in ζ, so that g_ζ is quasi-conformally conjugated to a map $f_{\lambda(\zeta)}$ with $\lambda(\zeta)$ continuous.

1. The map f_{λ_0}: Since no critical value is in T_{λ_0} ($\lambda_0 \notin \mathcal{H}_0$ and Lemma 1.3), $f_{\lambda_0}^{-1}(T_{\lambda_0})$ has exactly four connected components. One of them (at least) intersects the orbit of the critical points; we choose one that we call W_{λ_0}.

2. Definition of g_ζ: Let D be a disc in T_{λ_0} such that, for $\zeta \in D(\zeta_0, \epsilon)$, the Böttcher coordinate $\varphi_{\lambda_0}^{-1}(\zeta)$ is in D. Let D' be the preimage of D in W_{λ_0}. Outside D', iD', $-D'$ and $-iD'$, we take $g_\zeta = f_{\lambda_0}$ and inside D' we take for g_ζ a diffeomorphism sending $f_{\lambda_0}^{n-2}(1/4 + 4\lambda_0)$ to $\varphi_{\lambda_0}^{-1}(\zeta)$. On the symmetric domains iD', $-D'$ and $-iD'$, g_ζ is defined to reflect the symmetry of f_{λ_0} in the sense that $g_\zeta(-z) = g_\zeta(z)$ and $g_\zeta(iz) = -g_\zeta(z)$.

To construct g_ζ continuously in ζ starting from $g_{\zeta_0} = f_{\lambda_0}$ we take for the diffeomorphism g_ζ (in D') the composition of f_{λ_0} by a Möbius transformation (in the model of the round disc). More precisely, we can assume that $\varphi_{\lambda_0}^{-1}(D(\zeta_0, \epsilon)) \Subset D$ and take the Riemann map $\psi : (D, \varphi_{\lambda_0}^{-1}(\zeta_0)) \to (3\mathbf{D}, 0)$ sending $\varphi_{\lambda_0}^{-1}(\zeta_0)$ to 0 and $\varphi_{\lambda_0}^{-1}(\zeta)$ in $\mathbf{D}$. We define g_ζ in D' as $g_\zeta(z) = \psi^{-1} \circ M_\zeta \circ \psi \circ f_{\lambda_0}(z)$ where

$$M_\zeta(z) = \begin{cases} z & \text{for} \quad z \in 3\mathbf{D} \backslash 2\mathbf{D}\,; \\ \frac{z+a}{1+\bar{a}z} & \text{for} \quad z \in \mathbf{D} \quad \text{with } a = \psi \circ \varphi_{\lambda_0}^{-1}(\zeta)\,; \\ r\,\frac{e^{i\theta} + p(r)a}{1 + p(r)\bar{a}e^{i\theta}} & \text{for} \quad z = re^{i\theta} \in \mathbf{D} \backslash 2\mathbf{D}. \end{cases}$$

where $p : [1, 2] \to [0, 1]$ is a smooth map $p(1) = 1$, $p(2) = 0$ with all derivatives vanishing at those two points. Note that for $\zeta = \zeta_0$, the point a is 0 and $M_{\zeta_0} = id$. Hence $g_{\zeta_0} = f_{\lambda_0}$ and $g : D(\zeta_0, \epsilon) \times \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ defined by requiring $g_\zeta(z)$ is differentiable in (ζ, z).

3. *A complex structure invariant by g_ζ*: We define a complex structure σ_ζ of Beltrami coefficients μ_ζ as follows: Let μ_0 denote the Beltrami coefficient of the standard structure and set

$$\mu_\zeta(z)=\begin{cases}\mu_0 & \text{if} \quad z\in B_{\lambda_0};\\ (g_\zeta^n)^*(\mu_0) & \text{if} \quad g_\zeta^n(z)\in B_{\lambda_0};\\ \mu_0 & \text{if} \quad \notin A_{\lambda_0}\end{cases}$$

This complex structure is invariant by g_ζ and has the following properties:

- There exists $k<1$ such that $||\mu_\zeta||_\infty \le k$ for $\zeta\in D(\zeta_0,\epsilon)$. Indeed, g_ζ is holomorphic outside $i^k W_{\lambda_0}$ (with $k\in\mathbf{N}$), and g_ζ passes only once in $i^k W_{\lambda_0}$ where it is a diffeomorphism. Hence the dilatation $||\mu_\zeta||_\infty$ is bounded for each ζ and globally on $D(\zeta_0,\epsilon)$ by the continuity of $||\mu_\zeta||_\infty$ in ζ.

- The symmetries of g_ζ implies that $\mu_\zeta(-z)=\mu_\zeta(z)$ and $\mu_\zeta(iz)=\mu_\zeta(z)$ for $z\in\mathbf{C}$.

4. *The* map $f_{\lambda(\zeta)}$: By Ahlfors-Bers' Theorem, there exists a map $h_\zeta : \overline{\mathbf{C}}\to\overline{\mathbf{C}}$ such that $\mu_\zeta=h_\zeta^*\mu_0$ and h_ζ is continuous in ζ. Moreover h_ζ can be chosen to be tangent to identity at ∞ and fixing 0. Hence the symmetries are kept: $h_\zeta(-z)=-h_\zeta(z)$, $h_\zeta(iz)=ih_\zeta(z)$.

The map $R_\zeta=h_\zeta\circ g_\zeta\circ h_\zeta^{-1}$ is holomorphic (it preserves μ_0), and it can be written $R_\zeta(z)=z^2+az+b+\frac{cz+d}{z^2}$ since h_ζ preserves the pole 0 with multiplicity and the degree at ∞. From the symmetries of h_ζ it follows that $R_\zeta(-z)=R_\zeta(z)$ and $R_\zeta(iz)=-R_\zeta(z)$, and therefore $R_\zeta(z)=z^2+d/z^2$. This determines a unique and continuous $\lambda(\zeta)$ such that $R_\zeta=f_{\lambda(\zeta)}$. Indeed, R_ζ is continuous in ζ since g_ζ and h_ζ are continuous by construction and h_ζ^{-1} is also continuous: if $\zeta_i\to\zeta_\infty$ and $h_{\zeta_i}\to h_{\zeta_\infty}$, any accumulation value ζ of $h_{\zeta_i}^{-1}(z)$ on $\overline{\mathbf{C}}$ satisfies $h_{\zeta_\infty}(\zeta)=z$ i.e., $\zeta=h_{\zeta_\infty}^{-1}(z)$.

5. $\lambda(\zeta)$ is a section of Φ: On $B_{\lambda_0}\cup T_{\lambda_0}$ the map h_ζ is a holomorphic conjugacy between $f_{\lambda(\zeta)}$ and f_{λ_0}. Now the Böttcher coordinates on B_{λ_0} are related by $\varphi_{\lambda_0}=\varphi_{\lambda(\zeta)}\circ h_\zeta$ because they both conjugate f_{λ_0} to $z\mapsto z^2$ and have derivative 1 at ∞. On T_{λ_0} the maps $\widetilde{\varphi}_{\lambda_0}$ and $\widetilde{\varphi}_{\lambda(\zeta)}\circ h_\zeta$ can differ only by multiplication by -1. But $g_{\zeta_0}=f_{\lambda_0}$ so $h_{\zeta_0}=id$ and $R_{\zeta_0}=f_{\lambda_0}$, in particular $\lambda(\zeta_0)=\lambda_0$. Therefore $\widetilde{\varphi}_{\lambda(\zeta_0)}\circ h_{\zeta_0}$ and $\widetilde{\varphi}_{\lambda_0}$ coincide since they both have derivative $1/\sqrt{\lambda_0}$ at 0. Finally they coincide for every $\zeta\in D(\zeta_0,\epsilon)$ since the derivative at 0 is continuous.

Now we can evaluate $\Phi(\lambda(\zeta))=\widetilde{\varphi}_{\lambda(\zeta)}(f_{\lambda(\zeta)}^{n-1}(1/4+4\lambda(\zeta)))$. First, the critical points of $f_{\lambda(\zeta)}$ correspond to those of g_ζ under h_ζ, since it is a conjugacy. The critical points of g_ζ are those of f_{λ_0}, so their image by g_ζ^n in W_{λ_0} is $f_{\lambda_0}^{n-2}(1/4+4\lambda_0)$ and by definition of g_ζ the image in T_{λ_0} is $\tilde{\varphi}_{\lambda_0}^{-1}(\zeta)$. Hence $f_{\lambda(\zeta)}^{n-1}(1/4+4\lambda(\zeta))=h_\zeta(\tilde{\varphi}_{\lambda_0}^{-1}(\zeta))$.

Finally, $\Phi(\lambda(\zeta))=\widetilde{\varphi}_{\lambda(\zeta)}(f_{\lambda(\zeta)}^{n-1}(1/4+4\lambda(\zeta)))=\widetilde{\varphi}_{\lambda(\zeta)}(h_\zeta(\tilde{\varphi}_{\lambda_0}^{-1}(\zeta)))=\tilde{\varphi}_{\lambda_0}(\tilde{\varphi}_{\lambda_0}^{-1}(\zeta))=\zeta$. □

3. Counting the capture zones. We deduce from Proposition 2.6 that there are as many connected components of $\mathcal{H}_n$ ($n\neq 0$) as there are solutions of the equation $f_\lambda^{n-1}(1/4+4\lambda)=0$ (the centers). We determine this number now.

Corollary 3.1. *The roots of* $f_\lambda^{n-1}(1/4+4\lambda)$ *are simple.*

Proof. Let λ_0 be a solution of $f_\lambda^{n-1}(1/4+4\lambda)=0$ and assume, to arrive at a contradiction, that $f_\lambda^{n-1}(1/4+4\lambda) = a(\lambda-\lambda_0)^k + o((\lambda-\lambda_0)^k)$ for λ near λ_0 and some $k>1$. Since λ_0 belongs to $\mathcal{H}_n$, the map $(\lambda, z) \to \widetilde{\varphi}_\lambda(z)$ is well defined for (λ, z) near $(\lambda_0, 0)$ and holomorphic (Lemma 2.1). Its expansion near $(\lambda_0, 0)$ is $\widetilde{\varphi}_\lambda(z) = z/\sqrt{\lambda_0} + a(z, \lambda-\lambda_0)$, since $\widetilde{\varphi}_\lambda(0)=0$ for every λ near λ_0 (and Lemma 2.3). Hence $\Phi(\lambda) = \widetilde{\varphi}_\lambda(f_\lambda^{n-1}(1/4+4\lambda)) = a(\lambda-\lambda_0)^k/\sqrt{\lambda_0} + a\ (\lambda-\lambda_0)$ near λ_0, but this implies that $\Phi(\lambda)$ has a critical point at λ_0 and contradicts Proposition 2.6. □

Remark 3.2. There are exactly 4^{n-1} solutions of the equation $f_\lambda^{n-1}(1/4+4\lambda)=0$.

Proof. The rational map $f_\lambda^n(1/4+4\lambda)$ can be written in the form $P_n(\lambda)/Q_n(\lambda)$ with P_n and Q_n coprime polynomials. They are determined by $P_0(\lambda) = 1/4+4\lambda$, $Q_0(\lambda)=1$ and the relations $P_{n+1}(\lambda) = P_n(\lambda)^4 + \lambda Q_n(\lambda)^4$ and $Q_{n+1}(\lambda) = P_n(\lambda)^2 Q_n(\lambda)^2$. Note that P_{n+1} and Q_{n+1} are still coprime since $P_n(0) \neq 0$. Therefore, $\deg(P_n) > \deg(Q_n) + 1$ for all $n \geq 0$ and $\deg(P_n) = 4^n$. □

Corollary 3.3. *There are exactly* 4^{n-1} *component of* $\mathcal{H}_n$ *for* $n>0$.

4. Landing points of rays in the capture zones. Using the parametrization of the capture zones, one can define rays and equipotentials in a component $\mathcal{U}$ of $\mathcal{H}_n$ just by pulling back the rays and circles of **D** via the isomorphism Φ.

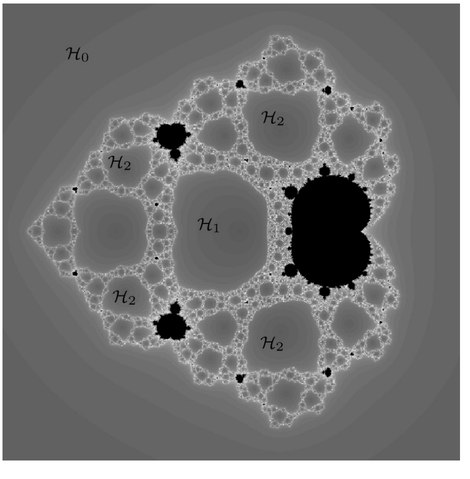


Figure 2: The connected components of $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$.

Lemma 4.1. *Let λ_0 be a parameter in the boundary of a capture zone $\mathcal{U}$ but not of $\mathcal{H}_0$. If λ_0 is the landing point of a ray $\mathcal{R}_{\mathcal{U}}(\theta)$ with $\theta \in \mathbf{Q}$ then λ_0 is a Misiurewicz parameter (i.e., the critical points of f_{λ_0} are eventually periodic).*

Proof. Let n be the order of $\mathcal{U}$. For $\lambda \in \mathcal{U}$, let U_λ be a component containing c_λ one of the critical points and denote by r_λ the center of U_λ i.e., the preimage of ∞ by f_λ^n. The points r_λ and c_λ can be followed continuously in $\mathcal{U}$ and r_λ admits a limit at λ_0 denoted by r_0. We denote by $R_{r_\lambda}(\theta)$ the ray of angle θ in U_λ, it is defined as the inverse image by f_λ^n of $R_\lambda(\theta) = \varphi_\lambda^{-1}(]1, +\infty[e^{2i\pi\theta})$ stemming from r_λ.

The ray $R_{r_0}(\theta)$ converges to a point $z_0 \in \partial U_{\lambda_0}$ which is eventually periodic. Indeed, the angle is in $\mathbf{Q}$ and the critical points are not in A_{λ_0} so the ray converges to a periodic or eventually periodic point of ∂U_{λ_0}(see [D-H]). If the landing point is periodic, its iterated images are on ∂U_{λ_0} but they are also eventually on ∂B_{λ_0}. This is not possible since it would imply that there is a ray in T_{λ_0} and a ray in B_{λ_0} converging to the same point and with image the same ray in B_{λ_0}, so the common landing point should be critical and this contradicts the fact that it is periodic and on the Julia set.

For $\lambda \in \mathcal{R}_{\mathcal{U}}(\theta)$ the ray $R_{r_\lambda}(\theta)$ contains the critical point c_λ (by definition). Hence, there is no stability properties of the ray $R_{r_0}(\theta)$ for λ near λ_0, so the landing point z_0 is necessarily parabolic or Misiurewicz (see [D-H]).

Assume now that z_0 is eventually parabolic. Then $1/4 + 4\lambda_0$ would be attracted by the parabolic point. But $1/4 + 4\lambda_0$ is in $\overline{f_{\lambda_0}^2(U_{\lambda_0})}$ since for $\lambda \in U$, $1/4 + 4\lambda \in \overline{f_\lambda^2(U_\lambda)}$ and $\overline{f_\lambda^2(U_\lambda)}$ is obtained by holomorphic motion from $\overline{f_{\lambda_0}^2(U_{\lambda_0})}$ in a neighborhood of λ_0. Indeed, on a neighborhood $D(\lambda_0, \epsilon)$ disjoint from $\overline{\mathcal{H}_0}$, the map $\varphi_\lambda^{-1}(z)$ defines a holomorphic motion of B_{λ_0}, by the Λ-Lemma it can be extended to $\overline{B}_{\lambda_0}$ and it can be pulled back to a holomorphic motion of $\overline{f_{\lambda_0}^2(U_{\lambda_0})}$.

Acknowledgements. I would like to thank R. Devaney for his comments and C. Henriksen for his remarks and a carefull reading of this text.

References

[B] P. Blanchard, *Complex analytic dynamics on the Riemann sphere*, Bull. Amer. Math. Soc. **11** (1984), 85–141

[B-D et al.] P. Blanchard, R. L. Devaney, D. M. Look, M. M. Rocha, P. Seal, D. Uminsky, *Sierpinski Carpets and Gaskets as Julia sets of rational Maps* in this volume.

[D-L] R. L. Devaney, D. M. Look, *A Criterion for Sierpinski Curve Julia Sets for Rational Maps*, *To appear in Topology Proceedings* 11 (2005), 581–596.

[D-H] A. Douady, J. H. Hubbard, *Etude dynamique des polynômes complexes*, Publications mathématiques d'Orsay 1984.

[F] D. Faught, *Local connectivity in a family of cubic polynomials*, Ph.D Thesis, Cornell, 1992.

[McM] C. T. McMullen, *Automorphisms of Rational Maps*, Holomorphic Functions and Moduli. Vol. 1. Math. Sci. Res. Inst. Publ. **10**. Springer, New York, 1988.

[M1] J. MILNOR, *Dynamics in One Complex Variable*, Published by Vieweg 1999, 2nd edition 2000.
[M2] J. MILNOR, *Hyperbolic components in spaces of polynomial maps*, IMS preprint 1992.

Laboratoire Paul Painlevé, UMR 8524
Université des Sciences et Technologies de Lille
Cité Scientifique
59655 Villeneuve d'Ascq Cedex, France
e-mail: roesch@math.univ-lille1.fr

Semiconjugacies between the Julia sets of geometrically finite rational maps II

Tomoki Kawahira

Abstract. For a geometrically finite rational map f, there exists a perturbation into another geometrically finite rational map which preserves the dynamics on the Julia set nicely. Indeed, we can perturb parabolic cycles of f into attracting cycles and repelling cycles in any combination.

1. Introduction. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. We call such a map *geometrically finite* if all critical points contained in the Julia set $J(f)$ are eventually periodic.

Perturbations of f. Let us fix a small $\epsilon_0 > 0$ and consider a family of rational maps of degree d, $\{f_\epsilon : \epsilon \in [0, \epsilon_0]\}$ with $f_0 = f$ and $f_\epsilon \to f$ uniformly and continuously with respect to the spherical metric. We represent this family in the convergence form, $f_\epsilon \to f$, and call it a *perturbation* of f.

From the viewpoint of J-stability, a good perturbation $f_\epsilon \to f$ is accompanied by a conjugacy $h_\epsilon : J(f_\epsilon) \to J(f)$ for each $\epsilon \in [0, \epsilon_0]$ with $h_\epsilon^{-1} \to \mathrm{id}$ as $\epsilon \to 0$. (That is, the dynamics on the Julia set is continuously perturbed.) In [3], the author gave some conditions for this as described later in §3. In particular, it is shown that even if a perturbation of f is not accompanied by such a conjugacy, it may be accompanied by a semiconjugacy (that is, continuous and surjective map $h_\epsilon : J(f_\epsilon) \to J(f)$ with $h_\epsilon \circ f_\epsilon = f \circ h_\epsilon$) which still preserves the dynamics of the Julia set except on a countable subset.

In this paper, following the result of [3] and as an application of Shishikura's perturbation in [5], we establish:

Theorem 1.1. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a geometrically finite rational map of degree d with $J(f) \neq \hat{\mathbb{C}}$. Then there exists a perturbation $f_\epsilon \to f$ such that*

(1) *f_ϵ is also geometrically finite;*
(2) *One can choose the direction of the perturbation such that the parabolic cycles of f are perturbed into repelling, parabolic and attracting cycles of f_ϵ in any combination;*
(3) *For each ϵ which is sufficiently small, there exists a unique semiconjugacy $h_\epsilon : J(f_\epsilon) \to J(f)$ with* $\sup\left\{d_{\hat{\mathbb{C}}}(h_\epsilon(x), x) : x \in J(f_\epsilon)\right\} \to 0$ $(\epsilon \to 0)$*; and*

(4) *If* card $(h_\epsilon^{-1}(y)) \geq 2$ *for some* $y \in J(f)$*, then there exists an* n *such that* $f^n(y)$ *is a parabolic periodic point of* f *and* $\mathrm{card}(h_\epsilon^{-1}(y)) = \deg(f^n, y) \cdot p(f^n(y))$*, where* $\deg(f^n, y)$ *is the local degree of* f^n *at* y*, and* $p(f^n(y))$ *is the number of attracting petals at* $f^n(y)$*.*

(5) h_ϵ *is a homeomorphism (thus conjugacy) between the Julia sets if and only if none of parabolic cycles of* f *is perturbed into an attracting cycle.*

The proof is given in the following sections.

Notes.

1. The case when $J(f) = \hat{\mathbb{C}}$ is somehow easier, since f is postcritically finite. By Thurston's theorem, if the orbifold of f is not of type (2, 2, 2, 2), there are only trivial perturbations by Möbius conjugations.
2. Property (3) of the theorem guarantees continuity of the Julia set along the perturbation with respect to the Hausdorff topology.
3. Property (4) of the theorem implies that the injectivity of h_ϵ may break only at countably many points. (We may say "h_ϵ is almost bijective".)
4. Our theorem generalize the results of [2]. One can find a similar result with a sophisticated investigation in [1].

Notation. Here we list some notation.

- $A(f)$: the set of all parabolic periodic points of f.
- $C(f)$: the set of all critical points of f.
- $P(f) := \overline{\{f^n(c) : c \in C(f), n = 1, 2, \dots\}}$; the postcritical set of f.
- $CP(f) := C(f) \cup P(f)$.
- $n \gg 0$ means that $n > 0$ is sufficiently large.
- $\epsilon \ll 1$ means that $\epsilon > 0$ is sufficiently small.

2. Shishikura's perturbation. For a rational map of given degree, M. Shishikura developed the method of perturbation which makes indifferent cycles into attracting cycles, and gave a sharp estimate of the number of non-repelling cycles [5]. This method is well prepared for application, like this:

Proposition 1.1. *For geometrically finite rational map* f *with* $J(f) \neq \hat{\mathbb{C}}$*, there exists a perturbation* $f_\epsilon \to f$ *with properties* (1) *and* (2) *of the Theorem* 1.1.

Proof. The rest of this section is devoted for the proof of this proposition.

Assumptions. By the assumption of geometric finiteness of f and $J(f) \neq \hat{\mathbb{C}}$, there is at least one attracting or parabolic cycle. By taking a Möbius conjugacy, we may also assume that ∞ is in the Fatou set and non-periodic. (We will add more precise condition later.)

Polynomial η. Let M be the maximal number of attracting petals over all $z \in A(f)$. (That is, the maximal multiplicity of parabolic cycles is $M+1$.) We take a monic polynomial $\eta(z)$ of degree $k \gg 0$ such that

- $\eta(z) = 0$ when z is in $CP(f) \cap J(f) - A(f)$ or in an attracting cycle; and
- for a parabolic cycle $\alpha = \{z_1, \ldots, z_p\} \subset A(f)$ with m (attracting) petals,
  - $\eta(z_i) = 0$;
  - $\eta'(z_i) = 1, 0,$ or -1 according to whether we want to perturb α into a repelling, parabolic, or attracting cycle respectively; and
  - $\eta^{(j)}(z_i) = 0$ for $2 \leq j \leq M+1$.

Thus η has an expansion of the form

$$\eta(z) = \sigma_\alpha(z - z_i) + a_{i,\,M+2}(z - z_i)^{M+2} + \cdots + (z - z_i)^k$$

about each $z_i \in \alpha$, where $\sigma_\alpha = \eta'(z_i)$.

Quasiregular map g_ϵ. Let $\rho : [0, \infty) \to [0, 1]$ be a smooth non-increasing function such that $\rho(t) = 1$ when $t \in [0, 1]$ and $\rho(t) = 0$ when $t \in [2, \infty)$. In particular, we can take such a ρ with bounded derivative. Set $H_\epsilon(z) = z + \epsilon\eta(z)\rho(\epsilon^{1/k}|z|)$. Then $H_\epsilon : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal map if $\epsilon \ll 1$, and satisfies $H_\epsilon \to \mathrm{id}$ and its maximum dilatation tends to 0 as $\epsilon \to 0$.

Next we set $g_\epsilon := f \circ H_\epsilon$. Then

- $g_\epsilon : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiregular map of degree d with $g_\epsilon \to f$;
- $g_\epsilon(z) = f(z)$ at z in an attracting or parabolic cycle, or in $CP(f) \cap J(f)$; and
- $\deg(g_\epsilon, c) = \deg(f, c)$ at $c \in C(f) \cap J(f)$.

Perturbation of non-repelling cycles. For each attracting or parabolic cycle $\alpha = \{z_1, \ldots, z_p\}$, we define two open sets $E(\alpha)$ and $E_\epsilon(\alpha)$ with $f(E(\alpha)) \subset E(\alpha)$ and $g_\epsilon(E_\epsilon(\alpha)) \subset E_\epsilon(\alpha)$ as following.

Attracting case. When α is attracting, we define $E(\alpha)$ by a disjoint union of p small topological disks near α with $f(\overline{E(\alpha)}) \subset E(\alpha)$. Set $E_\epsilon(\alpha) := E(\alpha)$. Then $g_\epsilon(\overline{E_\epsilon(\alpha)}) \subset E_\epsilon(\alpha)$ for $\epsilon \ll 1$ since g_ϵ converges uniformly to f.

Parabolic case. When α is parabolic with m (attracting) petals, we first consider the local dynamics near z_1. Set $\lambda := (f^p)'(z_1)$. (Then $\lambda^m = 1$.) It is known that we can take a conformal map $z = \psi(\zeta)$ defined near $\zeta = 0$ with $z_1 = \psi(0)$ and

$$F(\zeta) := \psi^{-1} \circ f^p \circ \psi(\zeta) = \lambda\zeta\{1 - \zeta^m + O(\zeta^{m+1})\}.$$

Fix an $r > 0$ and set $E' := \{\zeta : 0 < |\zeta| < r, |\arg \zeta^m| < \pi/3\}$. Then one can show that $F(\overline{E'}) \subset E' \cup \{0\}$ if $r \ll 1$.

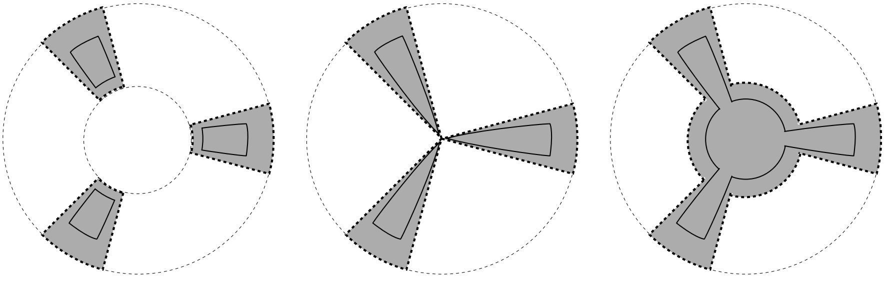


Figure 1: The case of $m = 3$. The shadowed region is E'_ϵ, with the boundary of $G_\epsilon(E'_\epsilon)$ approximately drawn in.

On the other hand, a basic calculation shows that

$$G_\epsilon(\zeta) := \psi^{-1} \circ g_\epsilon^p \circ \psi(\zeta) = \lambda\zeta\{(1+\sigma\epsilon)^p - \zeta^m + O(\epsilon\sigma\zeta) + O(\zeta^{m+1})\},$$

where $\sigma = \sigma_\alpha$. For $\epsilon \ll 1$, we define an open set E'_ϵ as following:

- If $\sigma = 1$, $E'_\epsilon := E' \cap \{|\zeta| > \epsilon^{2/(2m-1)}\}$;
- If $\sigma = 0$, $E'_\epsilon := E'$; and
- If $\sigma = -1$, $E'_\epsilon := E' \cup \{|\zeta| < \epsilon^{2/(2m-1)}\}$.

Then calculations as in [5, §4] show that if $\epsilon \ll 1$, $G_\epsilon(\overline{E'_\epsilon}) \subset E'_\epsilon$ when $\sigma = \pm 1$ and $G_\epsilon(\overline{E'_\epsilon}) \subset E'_\epsilon \cup \{0\}$ when $\sigma = 0$ (Figure 1).

Finally we set $E(\alpha) := \bigcup_{j=0}^{p-1} f^j(\psi(E'))$ and $E_\epsilon(\alpha) := \bigcup_{j=0}^{p-1} g_\epsilon^j(\psi(E'_\epsilon))$. Clearly they have the properties we desired.

Assumptions on infinity (Again). Set $E := \bigcup_\alpha E(\alpha)$ and $E_\epsilon := \bigcup_\alpha E_\epsilon(\alpha)$, where α ranges over all non-repelling cycles of f. Since there is at least one attracting or parabolic cycle in the Fatou set, E and E_ϵ are non-empty. Now we may assume that $\infty \in f^{-1}(E) - \overline{E}$ by taking a Möbius conjugacy. Then E and E_ϵ are uniformly bounded if $\epsilon \ll 1$. We have that $f(\infty) \in E$ and $g_\epsilon(\infty) \in E_\epsilon$ for all $\epsilon \ll 1$.

Getting rational perturbation. (See Lemma 3 of [5].) Now the quasiregular map g_ϵ is holomorphic except $V_\epsilon := \{z : |z| > \epsilon^{-1/k}\}$. Note that $f(V_\epsilon) \subset E$ and $g_\epsilon(V_\epsilon) \subset E_\epsilon$ if $\epsilon \ll 1$ by assumptions. Let σ_0 denote the standard complex structure on $\hat{\mathbb{C}}$. For $\epsilon \ll 1$, we put an almost complex structure σ_ϵ defined by $(g_\epsilon^n)^*(\sigma_0)$ on $g_\epsilon^{-n}(E_\epsilon)$ and by σ_0 otherwise. Then σ_ϵ is g_ϵ-invariant and we can find a quasiconformal map Φ_ϵ such that $\Phi_\epsilon^*\sigma_0 = \sigma_\epsilon$ a.e., and thus $f_\epsilon := \Phi_\epsilon \circ g_\epsilon \circ \Phi_\epsilon^{-1}$ is a rational map of degree d. Since $\Phi_\epsilon \to \mathrm{id}$ by the definition of g_ϵ and the continuous dependence of Φ_ϵ on its Beltrami differential, we obtain a perturbation $f_\epsilon \to f$ with (1) and (2) of Theorem 1.1 □

Remark.

- If f is a polynomial, $f_\epsilon \to f$ preserves the superattracting fixed point of degree d. Thus we actually have a polynomial perturbation.
- Perturbation of indifferent cycles as in [5, §4] does not guarantee continuity of the global dynamics in general. However, if f is geometrically finite, we have continuity of the dynamics at least on the Julia sets. (We need extra care of critical orbits, though.)

3. Existence of the semiconjugacy. Here we check that the perturbation $f_\epsilon \to f$ given in the previous section is accompanied by semiconjugacy as in Theorem 1.1. This is an application of a theorem from [3]. To state the theorem we introduce the notions of horocyclic perturbation and J-critical relations.

Horocyclic perturbation. We say a perturbation $f_\epsilon \to f$ is *horocyclic* if each parabolic periodic point a of f with m petals satisfies the following:

(a) If $f^p(a) = a$ and $(f^p)'(a) = \lambda$ (thus $\lambda^m = 1$), there exists a_ϵ with $f_\epsilon^p(a_\epsilon) = a_\epsilon \to a$, $(f_\epsilon^p)'(a_\epsilon) = \lambda_\epsilon \to \lambda$ as $\epsilon \to 0$;

(b) There is a neighborhood D of a with local coordinates $\phi_\epsilon, \phi : D \to \mathbb{C}$ such that:

1. $a_\epsilon \in D$ and $\phi_\epsilon(a_\epsilon) = \phi(a) = 0$;
2. $\phi_\epsilon \to \phi$ uniformly on D; and
3. If we represent the actions of f_ϵ^{pm} and f^{pm} on D by ϕ_ϵ and ϕ respectively, we obtain the local representation of the perturbation as:

$$\begin{aligned}\phi_\epsilon \circ f_\epsilon^{pm} \circ \phi_\epsilon^{-1}(z) &= \lambda_\epsilon^m z + z^{m+1} + O(z^{m+2}) \\ \longrightarrow \quad \phi \circ f^{pm} \circ \phi^{-1}(z) &= z + z^{m+1} + O(z^{m+2}). \qquad (3.1)\end{aligned}$$

In particular, ϕ, ϕ_ϵ are not necessarily conformal; they can be just homeomorphisms from D to their images.

(c) If we set $\exp(L_\epsilon + i\theta_\epsilon) := \lambda_\epsilon^m$, which tends to 1 as $\epsilon \to 0$, then $\theta_\epsilon^2 = o(|L_\epsilon|)$ as $L_\epsilon, \theta_\epsilon \to 0$.

Horocyclic perturbation was originally defined as *horocyclic convergence* of rational maps by C. McMullen [4, §7-9].

***J*-critical relations.** Let $c_1, \ldots, c_N$ be all critical points of f contained in *J(f)*, where N is counted *without* multiplicity. A *J-critical relation of f* is a set of non-negative integers *(i, j, m, n)* such that $f^m(c_i) = f^n(c_j)$.

We say a perturbation $f_\epsilon \to f$ *preserves the J-critical relations of f* if:

- For all $i = 1, \ldots, N$, the maps f_ϵ have critical points $c_i(\epsilon)$ (may be in the Fatou set) satisfying $c_i(\epsilon) \to c_i$ and $\deg(f_\epsilon, c_i(\epsilon)) = \deg(f, c_i)$ as $\epsilon \to 0$; and
- For each J-critical relation (i, j, m, n) of f, f_ϵ satisfies $f_\epsilon^m(c_i(\epsilon)) = f_\epsilon^n(c_j(\epsilon))$.

For such a perturbation $f_\epsilon \to f$, if f is geometrically finite, then the maps f_ϵ are also geometrically finite. If f is hyperbolic or parabolic, then $C(f) \cap J(f) = \emptyset$ and any small perturbation of f automatically preserves its J-critical relations.

Now we can state the following theorem:

Theorem 3.1 ([3]). *Let f be a geometrically finite rational map of degree ≥ 2, and $f_\epsilon \to f$ a horocyclic perturbation which preserves the J-critical relations of f. Then for each $\epsilon \ll 1$, we have a unique semiconjugacy $h_\epsilon : J(f_\epsilon) \to J(f)$ with properties* (3), (4) *and* (5) *of Theorem* 1.1.

Hence it is enough to show that $f_\epsilon \to f$ in the previous section is horocyclic and preserving the J-critical relations of f. In particular, since the J-critical relations are clearly preserved by the construction of f_ϵ, we only need to check each condition of horocyclic perturbation.

Condition (a) is clear by definition. Condition (c) is also easy since $\lambda_\epsilon^m = (1 \pm \sigma\epsilon)^{pm} \in \mathbb{R}$ and $\theta_\epsilon = 0$. For condition (b), since $\Phi_\epsilon \circ \psi$ converges to ψ uniformly near parabolic points of f, it is enough to consider $G_\epsilon \to F$ in the previous section. Now we complete the proof of Theorem 1.1 by claiming this:

Proposition 3.1. *By taking suitable local coordinates near* 0*, the convergence $G_\epsilon^m \to F^m$ can be viewed as in* (3.1).

Proof (See also Propositions 7.1 and 7.2 of [4].)**.** First let us check that the original $G_\epsilon^m \to F^m$ has the form

$$\begin{aligned} G_\epsilon^m(\zeta) &= \lambda_\epsilon^m \zeta + O(\sigma\epsilon\zeta^2) + C\zeta^{m+1} + O(\zeta^{m+2}) \text{ and} \\ &\longrightarrow F^m(\zeta) = \zeta + C\zeta^{m+1} + O(\zeta^{m+2}) \end{aligned}$$

with $C = -m$. For $j \geq 1$, we may set

$$\begin{aligned} G_\epsilon^j(\zeta) &= \lambda_\epsilon^j \zeta + O(\sigma\epsilon\zeta^2) + C'_j\zeta^{m+1} + O(\zeta^{m+2}) \text{ and} \\ F^j(\zeta) &= \lambda^j \zeta + C_j\zeta^{m+1} + O(\zeta^{m+2}) \end{aligned}$$

with $C'_1 = C_1 = -\lambda$. By comparing $G_\epsilon \circ G_\epsilon^j$ and $F \circ F^j$ with $G_\epsilon^j \circ G_\epsilon$ and F^{j+1} respectively, we have

$$C'_j = \lambda\lambda_\epsilon^{j-1} \cdot \frac{\lambda_\epsilon^{jm} - 1}{\lambda_\epsilon^m - 1} = -j\lambda^j(1 + O(\sigma\epsilon))$$

and $C_j = -j\lambda^j$. By putting m into j, we have the form of convergence as above. If $\sigma = 0$ then C can be normalized to be 1 by making a linear coordinate change $\zeta \mapsto Z = C^{1/m}\zeta$. Thus we consider the case of $\sigma = \pm 1$.

Now G_ϵ^m has the form

$$G_\epsilon^m(\zeta) = \lambda_\epsilon^m \zeta + A_\epsilon \zeta^N + O(\epsilon\zeta^{N+1}) + C\zeta^{m+1} + O(\zeta^{m+2})$$

where $2 \le N \le m$ and $A_\epsilon = O(\epsilon)$. Set $\Lambda_\epsilon := \lambda_\epsilon^m = 1 \pm mp\epsilon + O(\epsilon^2)$. Note that Λ_ϵ and Λ_ϵ are analytic functions of ϵ by the definitions of g_ϵ and G_ϵ. Consider a local coordinate

$$Z = \Psi_\epsilon(\zeta) = \zeta - B_\epsilon \zeta^N \text{ with } B_\epsilon = \frac{A_\epsilon}{\Lambda_\epsilon(\Lambda_\epsilon^{N-1} - 1)}.$$

Then Ψ_ϵ converges uniformly to another coordinate change $\Psi(\zeta) = \zeta - B\zeta^N$ near 0 as $\epsilon \to 0$. Let us check that $G_\epsilon^m \to F^m$ is locally represented as

$$\begin{aligned} \Psi_\epsilon \circ G_\epsilon^m \circ \Psi_\epsilon^{-1}(Z) &= \Lambda_\epsilon Z + O(\epsilon Z^{N+1}) + CZ^{m+1} + O(Z^{m+2}) \\ \longrightarrow \Psi \circ F^m \circ \Psi^{-1}(Z) &= Z + CZ^{m+1} + O(Z^{m+2}). \end{aligned}$$

One can easily check the form of $\Psi \circ F^m \circ \Psi^{-1}$. For $\Psi_\epsilon \circ G_\epsilon^m \circ \Psi_\epsilon^{-1}(Z)$, set $\tilde{G} := G_\epsilon^m$. Note that $\tilde{G}(\zeta) = \Lambda_\epsilon \zeta + O(\epsilon \zeta^N) + O(\zeta^{m+1})$ and thus

$$\begin{aligned} &\tilde{G}'(\zeta) = \Lambda_\epsilon + O(\epsilon \zeta^{N-1}) + O(\zeta^m); \\ &\tilde{G}^{(j)}(\zeta) = O(\epsilon) + O(\zeta^{m-j+1}) \text{ for } 2 \le j \le m; \text{ and} \\ &\tilde{G}^{(j)}(\zeta) = O(1) \text{ for } j \ge m+1. \end{aligned}$$

By a careful calculation, we have

$$\begin{aligned} \tilde{G}(\zeta) &= \tilde{G}'(Z + B_\epsilon \zeta^N) \\ &= \tilde{G}(Z) + \tilde{G}'(Z) B_\epsilon \zeta^N + \cdots + \frac{\tilde{G}^{(j)}(Z)}{j!}(B_\epsilon \zeta^N)^j + \cdots \\ &= \Lambda_\epsilon Z + A_\epsilon Z^N + \Lambda_\epsilon B_\epsilon \zeta^N + O(\epsilon Z^{N+1}) + CZ^{m+1} + O(Z^{m+2}) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \tilde{G}(\zeta)^N &= (\Lambda_\epsilon \zeta)^N \big(1 + O(\epsilon \zeta^{N-1}) + O(\zeta^m)\big)^N \\ &= \Lambda_\epsilon^N \zeta^N + O(\epsilon \zeta^{2N-1}) + O(\zeta^{m+N}) \\ &= \Lambda_\epsilon^N \zeta^N + O(\epsilon Z^{N+1}) + O(Z^{m+2}). \end{aligned} \tag{3.3}$$

By (3.2) and (3.3), we have

$$\begin{aligned} \Psi \circ \tilde{G}(\zeta) &= \tilde{G}(\zeta) - B_\epsilon \tilde{G}(\zeta)^N \\ &= \Lambda_\epsilon Z + A_\epsilon (Z^N - \zeta^N) + O(\epsilon Z^{N+1}) + CZ^{m+1} + O(Z^{m+2}). \end{aligned}$$

Since $Z^N - \zeta^N = O(\zeta^{2N-1}) = O(Z^{N+1})$ and $A_\epsilon = O(\epsilon)$, we have the form of $\Psi_\epsilon \circ G_\epsilon^m \circ \Psi_\epsilon^{-1}(Z) = \Psi \circ \tilde{G}(\zeta)$ as desired.

We can iterate this coordinate change until the coefficient of Z^m vanishes, and final linear coordinate changes normalize the coefficients of Z^{m+1} to be 1. Now we obtain the convergence of the form (3.1). □

Acknowledgement. I would like to thank M. Shishikura for suggesting this problem of perturbation and useful correspondence. I would also like to thank the referee for helpful suggestions and comments. This research is partially supported by JSPS Research Fellowships for Young Scientists.

References

[1] G. CUI, Geometrically finite rational maps with given combinatorics, *preprint*, 1997.

[2] P. HAÏSSINSKY, Déformation J-équivalence de polynômes géométriquement finis. *Fund. Math.* **163** (2000), no. 2, 131–141.

[3] T. KAWAHIRA, Semiconjugacies between the Julia sets of geometrically finite rational maps, *Erg. Th. & Dyn. Sys.* **23**(2003), 1125–1152.

[4] C. MCMULLEN, Hausdorff dimension and conformal dynamics II: Geometrically finite rational maps. *Comm. Math. Helv.* **75** (2000), no.4, 535–593

[5] M. SHISHIKURA, On the quasiconformal surgery of rational functions. *Ann. Sci. Éc. Norm. Sup.* **20** (1987), 1–29.

Graduate School of Mathematics
Nagoya University
Chikusa-ku, Nagoya
464-8602 Japan
e-mail: kawahira@math.nagoya-u.ac.jp

Homeomorphisms of the Mandelbrot Set

Wolf Jung

Abstract. On subsets of the Mandelbrot set, $\mathcal{E}_\mathcal{M} \subset \mathcal{M}$, homeomorphisms are constructed by quasi-conformal surgery. When the dynamics of quadratic polynomials is changed piecewise by a combinatorial construction, a general theorem yields the corresponding homeomorphism $h : \mathcal{E}_\mathcal{M} \to \mathcal{E}_\mathcal{M}$ in the parameter plane. Each h has two fixed points in $\mathcal{E}_\mathcal{M}$, and a countable family of mutually homeomorphic fundamental domains. Possible generalizations to other families of polynomials or rational mappings are discussed.

The homeomorphisms on subsets $\mathcal{E}_\mathcal{M} \subset \mathcal{M}$ constructed by surgery are extended to homeomorphisms of $\mathcal{M}$, and employed to study groups of non-trivial homeomorphisms $h : \mathcal{M} \to \mathcal{M}$. It is shown that these groups have the cardinality of $\mathbb{R}$, and they are not compact.

1. Introduction. Consider the family of complex quadratic polynomials $f_c(z) := z^2 + c$. They are parametrized by $c \in \mathbb{C}$, which is at the same time the critical value of f_c, since 0 is the critical point. The filled Julia set $\mathcal{K}_c$ of f_c is a compact subset of the dynamic plane. It contains all $z \in \mathbb{C}$ that are not attracted to ∞ under the iteration of f_c, i.e., $f_c^n(z) \not\to \infty$. The global dynamics is determined qualitatively by the behavior of the critical point or critical value under iteration. E.g., by a classical theorem of Fatou, $\mathcal{K}_c$ is connected iff $f_c^n(c) \not\to \infty$, i.e., $c \in \mathcal{K}_c$. The Mandelbrot set $\mathcal{M}$ is a subset of the parameter plane, it contains precisely the parameters with this property. Although it can be defined by the recursive computation of the critical orbit, with no reference to the whole dynamic plane, most results on $\mathcal{M}$ are obtained by an interplay between both planes: starting with a subset $\mathcal{E}_M \subset \mathcal{M}$, employ the dynamics of f_c to find a common structure in $\mathcal{K}_c$ for all $c \in \mathcal{E}_M$. Then an analogous structure will be found in $\mathcal{E}_M$, i.e., in the parameter plane. This principle has various precise formulations. Most important is its application to external rays: these are curves in the complement of $\mathcal{K}_c$ (dynamic rays) or in the complement of $\mathcal{M}$ (parameter rays), which are labeled by an angle $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$. For rational angles $\theta \in \mathbb{Q}/\mathbb{Z}$, these rays are landing at special points in $\partial\mathcal{K}_c$ or $\partial\mathcal{M}$, respectively. See Section 2.1 for details. When rays are landing together, the landing point is called a pinching point. It can be used to disconnect $\mathcal{K}_c$ or $\mathcal{M}$ into well-defined components. The structure of $\mathcal{K}_c$, as described by these pinching points, can be understood dynamically, and then these results are transfered to the parameter plane, to understand the structure of $\mathcal{M}$.

Each filled Julia set $\mathcal{K}_c$ is completely invariant under the corresponding mapping f_c, and this fact explains the self-similarity of these sets. On the other hand, when the parameter c moves through the Mandelbrot set, the corresponding Julia sets undergo an infinite number of bifurcations. By the above principle, the local structure of $\mathcal{M}$ is

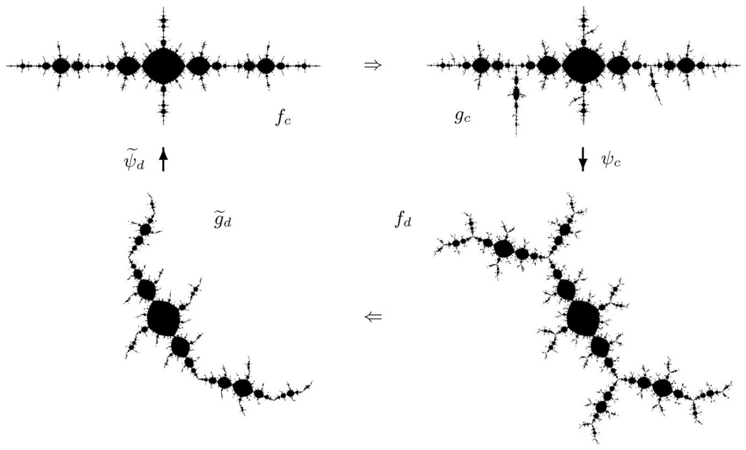


Figure 1: A simulation of Branner–Douady surgery $\Phi_A : \mathcal{M}_{1/2} \to \mathcal{T} \subset \mathcal{M}_{1/3}$, as explained in the text below. In this simulation, g_c and $\widetilde{g}_d$ are defined piecewise explicitly, and the required Riemann mappings are replaced with simple affine mappings.

undergoing corresponding changes as well. But these changes may combine in such a way, that subsets of $\mathcal{M}$ are mutually homeomorphic. Such homeomorphisms can be constructed by quasi-conformal surgery. There are three basic ideas (the first and second apply to more general situations [5]):

- A mapping g with desired dynamics is constructed piecewise, i.e., by piecing together different mappings or different iterates of one mapping. The pieces are defined, e.g., by dynamic rays landing at pinching points of the Julia set.
- g cannot be analytic, but one constructs a quasi-conformal mapping ψ such that the composition $f = \psi \circ g \circ \psi^{-1}$ is analytic. This is possible, when a field of infinitesimal ellipses is found that is invariant under g. Then ψ is constructed such that it is mapping these ellipses to infinitesimal circles. (See Section 2.2 for the precise definition of quasi-conformal mappings.)
- Suppose that f_c is a one-parameter family of analytic mappings, e.g., the quadratic polynomials, and that g_c is constructed piecewise from iterates of f_c for parameters $c \in \mathrm{e}_M \subset \mathcal{M}$. If $\psi_c \circ g_c \circ \psi_c^{-1} = f_d$, a mapping in parameter space is obtained from $h(c) := d$. Then one shows that h is a homeomorphism.

Homeomorphisms of the Mandelbrot set have been obtained in [6], [1], [2], [3], [14], [8]. We shall discuss the example of the Branner–Douady homeomorphism Φ_A, cf. Figure 1: parameters c in the limb $\mathcal{M}_{1/2}$ of $\mathcal{M}$ are characterized by the fact, that the filled Julia set $\mathcal{K}_c$ has two branches at the fixed point α_c and at its countable family of preimages. By a piecewise construction, f_c is replaced with a new mapping g_c, such that a third branch appears at α_c, and thus at its preimages as well. This can be done by cut- and paste techniques on a Riemann surface, or by conformal mappings between sectors in the dynamic plane. Since g_c is analytic except in some smaller sectors, it is possible to construct an invariant ellipse field. The corresponding quasi-conformal mapping ψ_c is used to conjugate g_c to a (unique) quadratic polynomial f_d, and the mapping in parameter space is defined by $\Phi_A(c) := d$. Now the Julia sets of f_d and g_c are homeomorphic, and the dynamics are conjugate. The parameter d belongs to the limb $\mathcal{M}_{1/3}$, since the three branches of $\mathcal{K}_d$ at α_d are permuted by f_d with rotation number $1/3$. Now there is an analogous construction of a mapping $\widetilde{g}_d$ for $d \in \mathcal{T} \subset \mathcal{M}_{1/3}$, which turns out to yield the inverse mapping $\widetilde{\Phi}_A$. The Julia set of $\widetilde{g}_d$ has lost some arms, and $\widetilde{g}_d$ is conjugate to a quadratic polynomial f_e again. By showing that f_c and f_e are conjugate, it follows that $e = c$, thus $\widetilde{\Phi}_A \circ \Phi_A = \mathrm{id}$. (The uniqueness follows from the fact that these quasi-conformal conjugations are hybrid-equivalences, i.e., conformal almost everywhere on the filled Julia sets [6].)

For the homeomorphisms constructed in this paper, the mapping g_c is defined piecewise by compositions of iterates of f_c, and no cut- and paste techniques or conformal mappings are used. Then the Julia sets of f_c and g_c are the same, and no arms are lost or added in the parameter plane either: a subset $\mathcal{E}_M \subset \mathcal{M}$ is defined by disconnecting $\mathcal{M}$ at two pinching points, and this subset is mapped onto itself by the homeomorphism (which is not the identity, of course). Thus a countable family of mutually homeomorphic subsets is obtained from one construction. General combinatorial assumptions are presented in Section 3.1, which allow the definition of a preliminary mapping $g_c^{(1)}$ analogous to the example in Figure 2: it differs from f_c on two strips V_c, W_c, where it is of the form

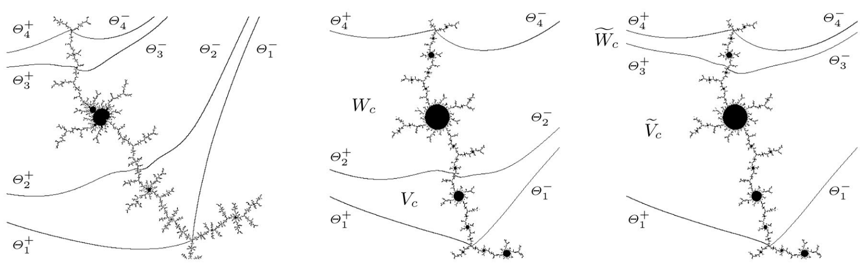


Figure 2: Application of Theorem 1.1. In this example, the subsets $\mathcal{E}_M \subset \mathcal{M}$ and $\mathcal{E}_c \subset \mathcal{K}_c$ are "edges" in the sense of Section 4.3. Left: a parameter edge $\mathcal{E}_M \subset \mathcal{M}$ and the strip $\mathcal{P}_M$. The homeomorphism h is mapping $\mathcal{E}_M$ to itself, cf. Figure 3. Middle and right: the dynamic edge $\mathcal{E}_c \subset \mathcal{K}_c$ in the strip $\overline{V_c \cup W_c} = \overline{\widetilde{V}_c \cup \widetilde{W}_c}$. According to Section 3.1, these strips are bounded by external rays, which belong to eight angles $\Theta_i^\pm$. Here we have $\Theta_1^- = 11/56$, $\Theta_2^- = 199/1008$, $\Theta_3^- = 103/504$, $\Theta_4^- = 23/112$, $\Theta_4^+ = 29/112$, $\Theta_3^+ = 131/504$, $\Theta_2^+ = 269/1008$, and $\Theta_1^+ = 15/56$. The first-return numbers are $k_w = \widetilde{k}_v = 4$, $k_v = \widetilde{k}_w = 7$. Now $g_c = g_c^{(1)}$ on $\mathcal{K}_c$ and the preliminary mapping is given by $g_c^{(1)} = f_c \circ \eta_c$, where η_c is mapping $\mathcal{E}_c$ to itself in such a way, that it is expanding in the lower strip and contracting in the upper strip. We have $\eta_c = f_c^{-2} \circ (-f_c^5) = f_c^{-3} \circ (+f_c^6) : V_c \to \widetilde{V}_c$, and $\eta_c = f_c^{-6} \circ (-f_c^3) : W_c \to \widetilde{W}_c$.

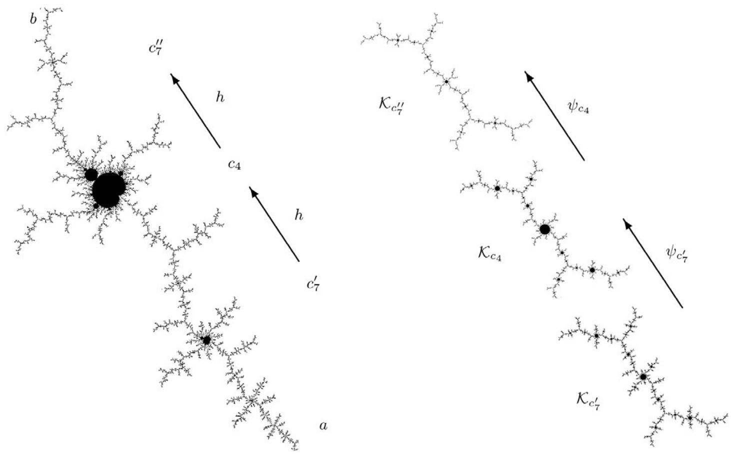


Figure 3: Left: the parameter edge $\mathcal{E}_\mathcal{M}$ from $a := \gamma_M(11/56)$ to $b := \gamma_M(23/112)$, the same as in Figure 2. The homeomorphism $h : \mathcal{E}_\mathcal{M} \to \mathcal{E}_\mathcal{M}$ is expanding at a and contracting at b. The centers of periods 4 and 7 are mapped as $h : c_7' \mapsto c_4 \mapsto c_7''$. Right: the filled Julia sets for c_7', c_4, and c_7'' are quasi-conformally homeomorphic. ($\mathcal{E}_c \subset \mathcal{K}_c$ is barely visible in the top right corner.)

$f_c^{-n} \circ (\pm\, f_c^m)$. Basically, we only need to find four strips with $\overline{V_c \cup W_c} = \overline{\widetilde{V}_c \cup \widetilde{W}_c}$, such that these are mapped as $V_c \to \widetilde{V}_c$, $W_c \to \widetilde{W}_c$ by suitable compositions of $\pm\, f_c^{\pm 1}$.

Theorem 1.1 (Construction and Properties of h)**.** *A subset $\mathcal{E}_M \subset \mathcal{M}$ is defined by its vertices, the Misiurewicz points a and b. For $c \in \mathcal{E}_M$, the preliminary mapping $g_c^{(1)}$ is constructed combinatorially: strips V_c and W_c are defined by external rays at suitable rational angles, such that the assumptions of Definition* 3.1 *on p. 7 are satisfied, and $g_c^{(1)}$ is defined by compositions of $\pm\, f_c^{\pm 1}$.*

1. *There is a family of "quasi-quadratic" mappings g_c coinciding with $g_c^{(1)}$ on the filled Julia sets $\mathcal{K}_c$. These are hybrid-equivalent to unique quadratic polynomials.*
2. *The mapping $h : \mathcal{E}_M \to \mathcal{E}_M$ in parameter space is defined as follows: for $c \in \mathcal{E}_M$, find the polynomial f_d that is hybrid-equivalent to g_c, and set $h(c) := d$. It does not depend on the precise choice of g_c (only on the combinatorial definition of $g_c^{(1)}$). Now h is a homeomorphism, and analytic in the interior of $\mathcal{E}_M$.*
3. *h is a non-trivial homeomorphism of $\mathcal{E}_M$ onto itself, fixing the vertices a and b. h and h^{-1} are Hölder continuous at Misiurewicz points and Lipschitz continuous at a and b. Moreover, h is expanding at a and contracting at b, cf. Figure* 3: *for $c \in \mathcal{E}_M \setminus \{a, b\}$ we have $h^n(c) \to b$ as $n \to \infty$, and $h^{-n}(c) \to a$. There is a countable family of mutually homeomorphic fundamental domains.*
4. *h extends to a homeomorphism between strips, $h : \mathcal{P}_M \to \widetilde{\mathcal{P}}_M$, which is quasi-conformal in the exterior of $\mathcal{M}$.*

The power of Theorem 1.1 lies in turning combinatorial data into homeomorphisms. The creative step remaining is to find eight angles $\Theta_i^\pm$, such that there are compositions of $\pm f_c^{\pm 1}$ mapping $V_c \to \widetilde{V}_c$ and $W_c \to \widetilde{W}_c$. When this is done, the existence of a corresponding homeomorphisms is guaranteed by the theorem.

In Sections 2 and 3, basic properties of $\mathcal{M}$ and of quasi-conformal mappings are recalled, and the proof of Theorem 1.1 is sketched by constructing g_c and h. Related results from the author's thesis [8] are summarized in Section 4. These include more examples of homeomorphisms, constructed at chosen Misiurewicz points or on "edges," and the combinatorial description of homeomorphisms. When a one-parameter family of polynomials is defined by critical relations, homeomorphisms in parameter space can be obtained by analogous techniques. The same applies e.g., to the rational mappings arising in Newton's method for cubic polynomials.

H. Kriete has suggested that the homeomorphisms $h : \mathcal{E}_M \to \mathcal{E}_M$ constructed by this kind of surgery extend to homeomorphisms of $\mathcal{M}$. Thus they can be used to study the homeomorphism group of $\mathcal{M}$, answering a question by K. Keller. In Section 5, some possible definitions for groups of non-trivial homeomorphisms are discussed, and their properties are obtained by combining two tools: the characterization of homeomorphisms by permutations of hyperbolic components, and the composition of homeomorphisms constructed by surgery. The groups are not compact, and the groups of non-trivial homeomorphisms have the cardinality of $\mathbb{R}$.

2. Background. Our main tools are the landing properties of external rays, and sending an ellipse field to circles by a quasi-conformal mapping.

2.1. The Mandelbrot Set. $f_c(z) = z^2 + c$ has a superattracting fixed point at $\infty \in \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. The unique Boettcher conjugation is conjugating f_c to $F(z) := z^2$, $\Phi_c \circ f_c = F \circ \Phi_c$ in a neighborhood of ∞. If the critical point 0, or the critical value c, does not escape to ∞, then $\mathcal{K}_c$ is connected [5], and the parameter c belongs to the Mandelbrot set $\mathcal{M}$ by definition. Then Φ_c extends to a conformal mapping $\Phi_c : \widehat{\mathbb{C}} \backslash \mathcal{K}_c \to \widehat{\mathbb{C}} \backslash \overline{\mathbb{D}}$, where $\overline{\mathbb{D}}$ is the closed unit disk. Dynamic rays $\mathcal{R}_c(\theta)$ are defined as preimages of straight rays $\mathcal{R}(\theta) = \{z \mid 1 < |z| < \infty,\ \arg(z) = 2\pi\theta\}$ under Φ_c. Theses curves in the complement of $\mathcal{K}_c$ may land at a point in $\partial\mathcal{K}_c$, or accumulate at the boundary without landing. But they are always landing when θ is rational. Each rational angle θ is periodic or preperiodic under doubling (mod 1). In the former case, the dynamic ray $\mathcal{R}_c(\theta)$ is landing at a periodic point $z = \gamma_c(\theta) \in \partial\mathcal{K}_c$, and at a preperiodic point in the latter case. These properties are understood from the relation $f_c(\mathcal{R}_c(\theta)) = \mathcal{R}_c(2\theta)$, which follows from $\arg(F(z)) = \arg(z^2) = 2\arg(z)$.

The Mandelbrot set is compact, connected, and full, and the conformal mapping $\Phi_M : \widehat{\mathbb{C}} \backslash \mathcal{M} \to \widehat{\mathbb{C}} \backslash \overline{\mathbb{D}}$ is given by $\Phi_M(c) := \Phi_c(c)$. (When $c \notin \mathcal{M}$, $\mathcal{K}_c$ is totally disconnected and Φ_c is not defined in all of its complement, but it is well-defined at the critical value.) Parameter rays $\mathcal{R}_M(\theta)$ are defined as preimages of straight rays under Φ_M. Their landing properties are obtained e.g., in [17]: each rational ray $\mathcal{R}_M(\theta)$ is landing at a point $c = \gamma_M(\theta) \in \partial\mathcal{M}$, and θ is called an external angle of c. When θ is preperiodic, then the critical value c of f_c is preperiodic, and the parameter c is called a Misiurewicz point. The critical value $c \in \mathcal{K}_c$ has the same external angles as the parameter $c \in \mathcal{M}$. When θ is periodic, then c is the root of a hyperbolic component (see below).

Both in the dynamic plane of f_c and in the parameter plane, the landing points of two or more rational rays are called pinching points. They are used to disconnect $\mathcal{K}_c$ or $\mathcal{M}$ into well-defined components, which are described combinatorially by rational numbers. Their structure is obtained from the dynamics, and transfered to the parameter plane. Pinching points with more than two branches are branch points.

Hyperbolic components of $\mathcal{M}$ consist of parameters, such that the corresponding polynomial has an attracting cycle. The root is the parameter on the boundary, such that the cycle has multiplier 1. The boundary of a hyperbolic component contains a dense set of roots of satellite components. Each hyperbolic component has a unique center, where the corresponding cycle is superattracting. Roots are dense at $\partial\mathcal{M}$, which itself is the accumulation set of centers in $\mathcal{M}$.

2.2. Quasi-Conformal Mappings. An orientation-preserving homeomorphism ψ between domains in $\widehat{\mathbb{C}}$ is K-quasi-conformal, $1 \le K < \infty$, if it has two properties:

- It is weakly differentiable, so that its differential $d\psi = \partial\psi\, dz + \overline{\partial}\psi\, d\overline{z}$ is defined almost everywhere. This linear map is sending certain ellipses in the tangent space to circles. The Beltrami coefficient $\mu := \overline{\partial}\psi / \partial\psi$ is defined almost everywhere. It encodes the direction and the dilatation ratio of the semi-axes for the ellipse field [5].
- The dilatation ratio is bounded globally by K, or $|\mu(z)| \le (K-1)/(K+1)$ almost everywhere.

The chain rule for derivatives is satisfied for the composition of quasi-conformal mappings, and a 1-quasi-conformal mapping is conformal. Quasi-conformal mappings are absolutely continuous, Hölder continuous, and have nice properties regarding, e.g., boundary behavior or normal families [11]. Given a measurable ellipse field (Beltrami coefficient) μ with $|\mu(z)| \le m < 1$ almost everywhere, the Beltrami differential equation $\overline{\partial}\psi = \mu\partial\psi$ on $\widehat{\mathbb{C}}$ has a unique solution with the normalization $\psi(z) = z + o(1)$ as $z \to \infty$. The dependence on parameters is described by the Ahlfors–Bers Theorem [5], which is behind some of our arguments, but will not be used explicitly here.

A *K-quasi-regular* mapping is locally K-quasi-conformal except for critical points, but it need not be injective globally. In Section 3.3, we will have a quasi-regular mapping g, such that all iterates are K-quasi-regular, and analytic in a neighborhood of ∞. Then a g-invariant field of infinitesimal ellipses is obtained as follows: it consists of circles in a neighborhood of ∞, i.e., $\mu(z) = 0$ there, and it is pulled back with iterates of g. Moreover, $\mu(z) := 0$ on the filled Julia set. Now ψ shall solve the corresponding Beltrami equation, i.e., send these ellipses to circles. By the chain rule, $f := \psi \circ g \circ \psi^{-1}$ is mapping almost every infinitesimal circle to a circle, thus it is analytic.

3. Quasi-Conformal Surgery. As soon as the combinatorial assumptions on $g_c^{(1)}$ given here are satisfied, Theorem 1.1 yields a corresponding homeomorphism of $\mathcal{E}_M$. After formulating these general assumptions, the proof is sketched by constructing the quasi-quadratic mapping g_c and the homeomorphism h. For some details, the reader will be referred to [8].

3.1. Combinatorial Setting. The following definitions may be illustrated by the example in Figure 2. Further examples are mentioned in Sections 4.2–4.4. When four parameter rays are landing in pairs at two pinching points of $\mathcal{M}$, this defines a strip in the parameter plane. Analogously, four dynamic rays define a strip in the dynamic plane. Our assumptions are formulated in terms of eight preperiodic angles

$$0 < \Theta_1^- < \Theta_2^- < \Theta_3^- < \Theta_4^- < \Theta_4^+ < \Theta_3^+ < \Theta_2^+ < \Theta_1^+ < 1. \tag{1}$$

- The Misiurewicz points $a := \gamma_M(\Theta_1^-) = \gamma_M(\Theta_1^+) \neq \gamma_M(\Theta_4^-) = \gamma_M(\Theta_4^+) =: b$ mark a compact, connected, full subset $\mathcal{E}_M \subset \mathcal{M}$; $\mathcal{E}_M = \mathcal{P}_M \cap \mathcal{M}$, where $\mathcal{P}_M$ is the closed strip bounded by the four parameter rays $\mathcal{R}_M(\Theta_1^\pm)$, $\mathcal{R}_M(\Theta_4^\pm)$.
- For all $c \in \mathcal{E}_M$, the eight dynamic rays $\mathcal{R}_c(\Theta_i^\pm)$ shall be landing in pairs at four distinct points, i.e., $\gamma_c(\Theta_i^-) = \gamma_c(\Theta_i^+)$. (Equivalently, they are landing in this pattern for one $c_0 \in \mathcal{E}_M$, and none of the eight angles is returning to $(\Theta_1^-,\ \Theta_1^+)$ under doubling mod 1.) Four open strips are defined as follows, cf. Figure 2: V_c is bounded by $\mathcal{R}_c(\Theta_1^\pm)$ and $\mathcal{R}_c(\Theta_2^\pm)$, W_c is bounded by $\mathcal{R}_c(\Theta_2^\pm)$ and $\mathcal{R}_c(\Theta_4^\pm)$, $\widetilde{V}_c$ is bounded by $\mathcal{R}_c(\Theta_1^\pm)$ and $\mathcal{R}_c(\Theta_3^\pm)$, $\widetilde{W}_c$ is bounded by $\mathcal{R}_c(\Theta_3^\pm)$ and $\mathcal{R}_c(\Theta_4^\pm)$. $\mathcal{E}_c \subset \mathcal{K}_c$ is defined as the intersection of $\mathcal{K}_c$ with the closed strip $\overline{V_c \cup W_c} = \overline{\widetilde{V}_c \cup \widetilde{W}_c}$. Thus for parameters $c \in \mathcal{E}_M$, the critical value c satisfies $c \in \mathcal{E}_c$.
- The first-return number k_v is the smallest integer $k > 0$, such that $f_c^k(V_c)$ meets (covers) $\mathcal{E}_c$. It is the largest integer $k > 0$, such that f_c^{k-1} is injective on V_c, since $\mathcal{E}_c$ contains the critical value c. Define $k_w, \widetilde{k}_v, \widetilde{k}_w$ analogously. They are independent of $c \in \mathcal{E}_M$. Now the main assumption on the dynamics, which makes finding the angles non-trivial, is that $f_c^{k_v-1}(V_c) = \pm f_c^{\widetilde{k}_v-1}(\widetilde{V}_c)$ and $f_c^{k_w-1}(W_c) = \pm f_c^{\widetilde{k}_w-1}(\widetilde{W}_c)$ for a choice of signs independent of c. The "orientation" is respected, i.e., with $z_i := \gamma_c(\Theta_i^\pm)$ we have e.g., $f_c^{k_v-1}(z_1) = \pm f_c^{\widetilde{k}_v-1}(z_1)$ and $f_c^{k_v-1}(z_2) = \pm f_c^{\widetilde{k}_v-1}(z_3)$.

If a or b is a branch point of $\mathcal{M}$, the last assumption implies that $\mathcal{E}_M$ is contained in a single branch, i.e., $\mathcal{E}_M \setminus \{a,\ b\}$ is a connected component of $\mathcal{M} \setminus \{a,\ b\}$.

Definition 3.1 (Preliminary Mapping $g_c^{(1)}$)**.** Under these assumptions, with the unique choices of signs in the two strips, define $\eta_c := f_c^{-(\widetilde{k}_v-1)} \circ (\pm f_c^{k_v-1}) : V_c \to \widetilde{V}_c$, $\eta_c := f_c^{-(\widetilde{k}_w-1)} \circ (\pm f_c^{k_w-1}) : W_c \to \widetilde{W}_c$, and $\eta_c := \mathrm{id}$ on $\mathbb{C} \setminus \overline{V_c \cup W_c}$ for $c \in \mathcal{E}_M$. Then define $g_c^{(1)} := f_c \circ \eta_c$ and $\widetilde{g}_c^{(1)} := f_c \circ \eta_c^{-1}$.

The three mappings are piecewise holomorphic, thus they cannot be extended continuously. Each has "shift discontinuities" on six dynamic rays: e.g., consider $z_0 \in \mathcal{R}_c(\Theta_2^-)$, $(z_n') \subset V_c$ and $(z_n'') \subset W_c$ with $z_n' \to z_0$ and $z_n'' \to z_0$, then $\lim g_c^{(1)}(z_n')$ and $\lim g_c^{(1)}(z_n'')$ both exist and belong to $\mathcal{R}_c(\Theta_3^-)$, but they are shifted relative to each other along this ray. Neglecting these rays, $g_c^{(1)}$ and $\widetilde{g}_c^{(1)}$ are proper of degree 2. In the following section, $g_c^{(1)}$ will be replaced with a smooth mapping g_c, which is used to construct the homeomorphism h. Analogously, $\widetilde{g}_c^{(1)}$ yields $\widetilde{h} = h^{-1}$.

3.2. Construction of the Quasi-Quadratic Mapping g_c. For $c \in \mathcal{E}_M$, we construct a quasi-regular mapping g_c coinciding with $g_c^{(1)}$ on $\mathcal{K}_c$. By employing the Boettcher conjugation Φ_c, the work will be done in the exterior of the unit disk $\overline{\mathbb{D}}$. This is convenient when $c \in \mathcal{E}_M$, and essential to construct the homeomorphism h in the exterior. In $\widehat{\mathbb{C}} \backslash \mathcal{K}_c$ we have $g_c^{(1)} = \Phi_c^{-1} \circ G^{(1)} \circ \Phi_c$, where $G^{(1)} : \widehat{\mathbb{C}} \backslash \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \backslash \overline{\mathbb{D}}$ is discontinuous on six straight rays, and given by compositions of $F(z) := z^2$ in the regions between these rays—it is independent of c in particular.

1. First construct smooth domains U, U' with $\overline{\mathbb{D}} \subset U$ and $\overline{U} \subset U'$, and a smooth mapping $G : U \backslash \overline{\mathbb{D}} \to U' \backslash \overline{\mathbb{D}}$. It shall be proper of degree 2 and coincide with $G^{(1)}$ except in sectors around those six rays $\mathcal{R}(\Theta_i^\pm)$, where $G^{(1)}$ has a shift discontinuity. The sectors are of the form $|\arg z - 2\pi\Theta_i^\pm| < s \log |z|$, and there G is a diffeomorphism. To ensure that the dilatation bound does not explode at the vertex of the sector, $\log G(z) - \mathrm{i}2\pi\Theta_j^\pm$ is chosen conveniently as a 1-homogeneous function of $\log z - \mathrm{i}2\pi\Theta_i^\pm$. (This simplifies the construction of [3], which employed a pullback of quadrilaterals.) The domains are chosen in a finite recursion, relying on the fact that some iterate of $G^{(1)}$ is strictly expanding [8, Section 5.2]. Since any orbit is visiting at most two of the sectors, the dilatation of all iterates of G is bounded uniformly.
2. Choose the radius $R > 1$ and the conformal mapping $H : \widehat{\mathbb{C}} \backslash \overline{U'} \to \widehat{\mathbb{C}} \backslash \overline{\mathbb{D}}_{R^2}$ with the normalization $H(z) = z + \mathcal{O}(1/z)$ at ∞ (which determines R and H uniquely). Extend H to a quasi-conformal mapping $H : \widehat{\mathbb{C}} \backslash U \to \widehat{\mathbb{C}} \backslash \mathbb{D}_R$ with $F \circ H = H \circ G$ on ∂U. Define the extended $G : \widehat{\mathbb{C}} \backslash \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \backslash \overline{\mathbb{D}}$ by $G := H^{-1} \circ F \circ H$ on $\widehat{\mathbb{C}} \backslash U$. Now G is proper of degree 2, quasi-regular, and the dilatation of G^n is bounded by some K uniformly in n. Finally, extend H to a mapping $H : \widehat{\mathbb{C}} \backslash \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \backslash \overline{\mathbb{D}}$ by recursive pullbacks, such that $F \circ H = H \circ G$ everywhere, then H is K-quasi-conformal. Cf. Figure 4.
3. Now, set $g_c := g_c^{(1)}$ on $\mathcal{K}_c$ and $g_c := \Phi_c^{-1} \circ G \circ \Phi_c$ on $\widehat{\mathbb{C}} \backslash \mathcal{K}_c$. Then g_c is a quasi-quadratic mapping, i.e., proper of degree 2, with a uniform bound on the dilatation of the iterates, with $\overline{\partial} g_c = 0$ a.e. on its filled Julia set $\mathcal{K}_c$, and analytic in a neighborhood of ∞ with $g_c(z) = z^2 + \mathcal{O}(1)$. (It is continuous at $\gamma_c(\Theta_i^\pm)$ by Lindelöf's Theorem.)

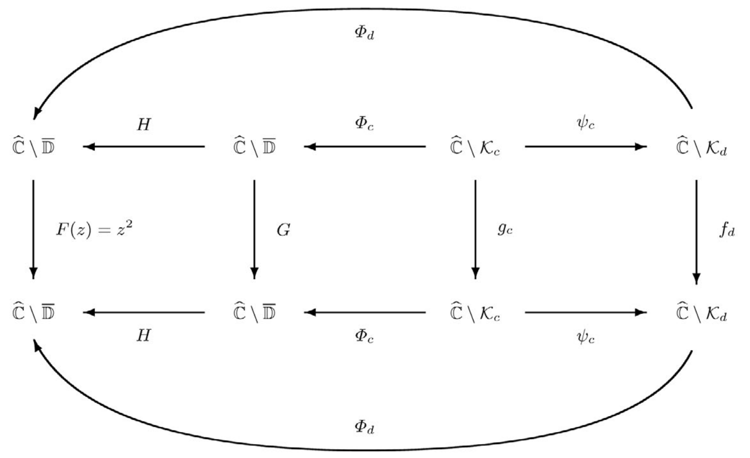


Figure 4: Construction and straightening of g_c by employing mappings in the exterior of the unit disk. (The hybrid equivalence ψ_c, which is conjugating g_c to a polynomial f_d, will be constructed in Section 3.3) If the filled Julia sets are not connected, the diagram is well-defined and commuting on smaller neighborhoods of ∞.

Now suppose that $c \in \mathcal{P}_M \setminus \mathcal{E}_M$ with $\Phi_M(c) \in U'$. Then $\mathcal{K}_c$ is totally disconnected, and Φ_c is not defined in all of $\widehat{\mathbb{C}} \setminus \mathcal{K}_c$. It can be defined, however, in a domain mapped to the six sectors and to $\widehat{\mathbb{C}} \setminus \overline{U}$ by Φ_c [3], [8]. Thus g_c is defined in this case as well, by matching $g_c^{(1)}$ with $\Phi_c^{-1} \circ G \circ \Phi_c$.

In the following section, we shall construct an invariant ellipse field for the quasi-quadratic g_c, and employ it to straighten g_c, i.e., to conjugate it to a quadratic polynomial f_d. Then we set $h(c) := d$. If we had skipped step 2, g_c would not be a quasi-quadratic mapping $\mathbb{C} \to \mathbb{C}$, but a quasi-regular quadratic-like mapping (cf. [6], [8]) between bounded domains $U_c \to U_c'$. This distinction is related to possible alternative techniques:

Remark 3.2 (Alternative Techniques)**.**

1. The classical techniques would be as follows [1], [2]: after the quasi-regular quadratic-like mapping $g_c : U_c \to U_c'$ is constructed, it is not extended to $\mathbb{C}$, but it is first conjugated to an analytic quadratic-like mapping, employing an invariant ellipse field in U_c'. Then the latter mapping is straightened to a polynomial by the Straightening Theorem [6]. With this approach, it will not be possible to extend the homeomorphism h to the exterior of $\mathcal{M}$.
2. Here we shall use the same techniques as in [3]: having extended g_c to $\widehat{\mathbb{C}}$, it will be easy to straighten. Instead of applying the Straightening Theorem, its proof [5] was adapted into the construction of g_c. This approach makes the extension of h to the exterior of $\mathcal{M}$ possible. By applying this technique to the construction of g_c and $h(c)$

for $c \in \mathcal{E}_M$ as well, the proofs of bijectivity, continuity, and of landing properties (Section 4.1) are simplified.

3. Alternatively, $g_c : U_c \to U'_c$ could be constructed as a quasi-regular quadratic-like mapping on a bounded domain, and be straightened without extending it to $\widehat{\mathbb{C}}$ first, by incorporating the alternative proofs of the Straightening Theorem according to [6]. This proof is more involved, but it has the advantage that the mapping H can be chosen more freely on $\overline{U'}\backslash U$, e.g., such that it is the identity on $\mathcal{R}(\Theta_1^\pm)$ and $\mathcal{R}(\Theta_4^\pm)$ [8]. Then h would be the identity on the corresponding parameter rays, which makes it easier to paste different homeomorphisms together.

3.3. *h* is a Homeomorphism. For $c \in \mathcal{E}_M$, or $c \in \mathcal{P}_M\backslash\mathcal{E}_M$ with $\Phi_M(c) \in U'$, the quasi-quadratic mapping g_c was constructed in the previous section. Now construct the g_c-invariant ellipse field μ by pullbacks with g_c, such that $\mu = 0$ in a neighborhood of ∞ and a.e. on $\mathcal{K}_c$. It is bounded by $(K-1)/(K+1)$, since the dilatation of all iterates g_c^n is bounded by K. Denote by ψ_c the solution of the Beltrami equation $\bar\partial\psi = \mu\partial\psi$, normalized by $\psi_c(z) = z + \mathcal{O}(1/z)$, which is mapping the infinitesimal ellipses described by μ to circles. Now $\psi_c \circ g_c \circ \psi_c^{-1}$ is analytic on $\widehat{\mathbb{C}}$ and proper of degree 2, thus a quadratic polynomial of the form $f_d(z) = z^2 + d$. In a neighborhood of ∞, $H \circ \Phi_c \circ \psi_c^{-1}$ is conjugating f_d to F, cf. Figure 4. By the uniqueness of the Boettcher conjugation, this mapping equals Φ_d. Recursive pullbacks show equality in $\widehat{\mathbb{C}}\backslash\mathcal{K}_d$, if $\mathcal{K}_c$ and $\mathcal{K}_d$ are connected, i.e., for $c \in \mathcal{E}_M$. Otherwise, equality holds on an f_d-forward-invariant domain of Φ_d, which may be chosen to include the critical value d.

We set $h(c) := d = \psi_c(c)$. If $c \in \mathcal{E}_M$, a combinatorial argument shows $d \in \mathcal{E}_M$. If $c \in \mathcal{P}_M\backslash\mathcal{E}_M$ with $\Phi_M(c) \in U'$, we have

$$\Phi_M(d) = \Phi_d(d) = \Phi_d \circ \psi_c(c) = H \circ \Phi_c(c) = H \circ \Phi_M(c). \tag{2}$$

Denote by $\widetilde{\mathcal{P}}_M$ the closed strip that is bounded by the four curves $\Phi_M^{-1} \circ H(\mathcal{R}(\Theta_i^\pm))$, $i = 1,\ 4$, which are quasi-arcs. Now h is extended to $h : \mathcal{P}_M \to \widetilde{\mathcal{P}}_M$ by setting

$$h := \Phi_M^{-1} \circ H \circ \Phi_M \ :\ \mathcal{P}_M\backslash\mathcal{E}_M \to \widetilde{\mathcal{P}}_M\backslash\mathcal{E}_M. \tag{3}$$

By (2), this agrees with the definition of $h(c)$ by straightening g_c, if $\Phi_M(c) \in U'$. Now (3) shows that h is bijective and K-quasi-conformal in the exterior of $\mathcal{E}_M$. We will see that h is bijective and continuous on $\mathcal{E}_M$. Let us remark that for $c \in \mathcal{E}_M$, the value of $d = h(c)$ does not depend on the choices made in the construction of G and H, since ψ_c is a hybrid-equivalence [6]. The proof of bijectivity in [2] relied on this independence, but the following one is simplified by employing H:

For $d \in \mathcal{E}_M$, consider $\widetilde{g}_d^{(1)}$ according to Definition 3.1, and define the quasi-quadratic mapping $\widetilde{g}_d$ with $\widetilde{g}_d := \widetilde{g}_d^{(1)}$ on $\mathcal{K}_d$ and $\widetilde{g}_d := \Phi_d^{-1} \circ \widetilde{G} \circ \Phi_d$ in $\widehat{\mathbb{C}}\backslash\mathcal{K}_d$, where $\widetilde{G} := H \circ F \circ H^{-1}$. To see that this choice is possible, note that H is mapping the region $V \subset U\backslash\overline{\mathbb{D}}$ (corresponding to V_c) to a distorted version of $\widetilde{V}$. There we have

$\widetilde{G}^{(1)} = F^{2-k_v} \circ (\pm \widetilde{F}^{k_v-1})$. Observing that $F = H \circ G \circ H^{-1}$ and H commutes with $\pm$ id on the set in question, we have $\widetilde{G}^{(1)} = H \circ G^{2-k_v} \circ (\pm \widetilde{G}^{k_v-1}) \circ H^{-1}$. Following the orbit and applying the piecewise definition of $G^{(1)}$ yields $\widetilde{G}^{(1)} = H \circ F \circ H^{-1}$. Together with the same result in other regions, this justifies the definition of $\widetilde{G}$, i.e., $\widetilde{g}_d$ is quasi-quadratic. Now $\widetilde{h}(d)$ is defined by straightening $\widetilde{g}_d$. —Suppose that $c \in \mathcal{E}_M$ and $d = h(c)$, then $f_d = \psi_c \circ g_c \circ \psi_c^{-1}$ and by its definition in terms of $H = \Phi_d \circ \psi_c \circ \Phi_c^{-1}$, we have $\widetilde{g}_d = \psi_c \circ f_c \circ \psi_c^{-1}$. Therefore $c = \widetilde{h}(d)$ and $\widetilde{\psi}_d = \psi_c^{-1}$. $\widetilde{h} \circ h = \text{id}$ and the converse result imply that $h : \mathcal{E}_M \to \mathcal{E}_M$ is bijective with $h^{-1} = \widetilde{h}$.

By (3), h is quasi-conformal in the exterior of $\mathcal{E}_M$. The interior of $\mathcal{E}_M$ consists of a countable family of hyperbolic components, plus possibly a countable family of non-hyperbolic components. The former are parametrized by multiplier maps, the latter by transforming invariant line fields. In both cases, h is given by a composition of these analytic parametrizations [2], [8]. It remains to show that h is continuous at $c_0 \in \partial\mathcal{E}_M$: suppose $c_n \to c_0$, $d_n = h(c_n)$, $d_0 = h(c_0)$. By bijectivity we have $d_0 \in \partial\mathcal{E}_M = \mathcal{E}_M \cap \partial\mathcal{M}$. It does not matter if c_n belongs to $\mathcal{E}_M$ or not. (Now we employ the definition of h by straightening g_c, which is equivalent to (3). If some $\gamma_c(\Theta_i^\pm)$ is iterated to $\gamma_c(\Theta_1^\pm)$ the case of $c_0 = \gamma_M(\Theta_i^\pm)$ requires extra treatment.) It is sufficient to show $d_n \to d_* \Rightarrow d_* = d_0$. Since the K-quasi-conformal mappings ψ_n are normalized, there is a K-quasi-conformal Ψ and a subsequence $\psi_{c_n'} \to \Psi$, uniformly on $\widehat{\mathbb{C}}$ [11]. We have $\psi_{c_n'} \circ g_{c_n'} \circ \psi_{c_n'}^{-1} \to \Psi \circ g_{c_0} \circ \Psi^{-1}$ and $\psi_{c_n} \circ g_{c_n} \circ \psi_{c_n}^{-1} = f_{d_n} \to f_{d_*}$, thus $\Psi \circ \psi_{c_0}^{-1}$ is a quasi-conformal conjugation from f_{d_0} to f_{d_*}. Although it need not be a hybrid-equivalence, $d_0 \in \partial\mathcal{M}$ implies $d_* = d_0$ [6]. By the same arguments, or by the Closed Graph Theorem, h^{-1} is continuous as well. Thus $h : \mathcal{P}_M \to \widetilde{\mathcal{P}}_M$ is a homeomorphism mapping $\mathcal{E}_M \to \mathcal{E}_M$.

3.4. Further Properties of *h*. Since h is analytic in the interior of e_M and quasi-conformal in the exterior, it is natural to ask if it is quasi-conformal globally. B. Branner and M. Lyubich are working on a proof employing quasi-regular quadratic-like germs. Maybe an alternative proof can be given by constructing a homotopy from f_c to $\widetilde{g}_d$, thus from id to h.

The dynamics of h on $\mathcal{E}_M$ is simple: set $c_0 := \gamma_M(\Theta_2^\pm)$ and $c_n := h^n(c_0)$, $n \in \mathbb{Z}$. The connected component of $\mathcal{E}_M$ between the two pinching points c_n and c_{n+1} is a fundamental domain for $h^{\pm 1}$. These domains are accumulating at the Misiurewicz points a and b, and the method of [18] yields a linear scaling behavior. Thus h and h^{-1} are Lipschitz continuous at a and b (and Hölder continuous at all Misiurewicz points). For $c \in \mathcal{E}_M \backslash \{a,\ b\}$ we have $h^n(c) \to b$ as $n \to \infty$ and $h^n(c) \to a$ as $n \to -\infty$.

4. Related Results and Possible Generalizations. Further results and examples from [8] are sketched, and some ideas on surgery for general one-parameter families are presented.

4.1. Combinatorial Surgery. The unit circle $\partial\mathbb{D}$ is identified with $S^1 := \mathbb{R}/\mathbb{Z}$ by the parametrization $\exp(i2\pi\theta)$. For h constructed from $g_c^{(1)}$ according to Theorem 1.1, recall the mappings $F, G, H : \widehat{\mathbb{C}}\setminus\overline{\mathbb{D}} \to \widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}$ from Section 3.2. Denote their boundary values by $\mathbf{F}, \mathbf{G}, \mathbf{H} : S^1 \to S^1$. Thus $\mathbf{F}(\theta) = 2\theta \bmod 1$ and $\mathbf{G}$ is piecewise linear. Now $\mathbf{H}$ is the unique orientation-preserving circle homeomorphism conjugating $\mathbf{G}$ to $\mathbf{F}$, $\mathbf{H} \circ \mathbf{G} \circ \mathbf{H}^{-1} = \mathbf{F}$. $\mathbf{H}(\theta)$ is computed numerically from the orbit of θ under $\mathbf{G}$ as follows: for $n \in \mathbb{N}$, the n-th binary digit of $\mathbf{H}(\theta)$ is 0 if $0 \le \mathbf{G}^{n-1}(\theta) < 1/2$, and 1 if $1/2 \le \mathbf{G}^{n-1}(\theta) < 1$. For rational angles, the (pre-) periodic sequence of digits is obtained from a finite algorithm.

In the exterior of $\mathcal{E}_M$, h is represented by H according to (3). Applying this formula to parameter rays and employing Lindelöf's Theorem shows: $\mathcal{R}_M(\theta)$ is landing at $c \in \partial\mathcal{E}_M$, iff $\mathcal{R}_M(\mathbf{H}(\theta))$ is landing at $h(c)$. If c is a Misiurewicz point or a root, then θ is rational, and $\mathbf{H}(\theta)$ is computed exactly. In this sense, $d = h(c)$ is determined combinatorially. Alternatively, one can construct the critical orbit of $g_c^{(1)}$ and the Hubbard tree of f_d. The simplest case is given when the critical orbit meets $\mathcal{E}_c$ only once: then the orbit of c under $g_c^{(1)}$ is the same as the orbit of $\eta_c(c)$ under f_c.

Regularity properties of $\mathbf{H}$ are discussed in [8, Section 9.2]. $\mathbf{H}$ has Lipschitz or Hölder scaling properties at all rational angles. $\mathbf{H}$ and $\mathbf{H}^{-1}$ are Hölder continuous with the optimal exponents $\widetilde{k}_v/k_v$ and $k_w/\widetilde{k}_w$. Since H is K-quasi-conformal, Mori's Theorem [11] says that $\mathbf{H}^{\pm 1}$ is $1/K$-Hölder continuous. Thus we have the lower bound $K \ge \max(k_v/\widetilde{k}_v,\ \widetilde{k}_w/k_w)$, independent of the choices made in the construction of $h : \mathcal{P}_M\setminus\mathcal{E}_M \to \widetilde{\mathcal{P}}_M\setminus\mathcal{E}_M$. By a piecewise construction we obtain a homeomorphism $h_* : \mathcal{M} \to \mathcal{M}$, which extends to a homeomorphism of $\mathbb{C}$, but such that no extension can be quasi-conformal.

4.2. Homeomorphisms at Misiurewicz Points. A homeomorphism $h : \mathcal{E}_M \to \mathcal{E}_M$ according to Theorem 1.1 is expanding at the Misiurewicz point a. Asymptotically, $\mathcal{M}$ shows a linear scaling behavior at a. (In Figure 3, you can observe the asymptotic self-similarity of $\mathcal{M}$ at a, and similarity between $\mathcal{M}$ at $c \approx c_n$ and $\mathcal{K}_{c_n}$ at $z \approx 0$.) Now h is asymptotically linear in a "macroscopic" sense, e.g., there is an asymptotically linear sequence of fundamental domains, but this is not true pointwise. These results are obtained by combining the techniques from [18] with the combinatorial description of h according to Section 4.1: consider a suitable sequence $c_n \to a$. If the critical orbit of f_{c_n} travels through $\mathcal{E}_{c_n}$ once, then h is asymptotically linear on the sequence, but it is not if the orbit meets $\mathcal{E}_{c_n}$ twice.

Conversely, given a branch at some Misiurewicz point a, is there an appropriate homeomorphism h? We only need to find a combinatorial construction of $g_c^{(1)}$. This was done in [8] for all β-type Misiurewicz points. (Here $\mathcal{P}_M$ and $\overline{V_c \cup W_c}$ are sectors, not strips.) The result is extended to all Misiurewicz points in [9]. The author's research was motivated by discussions with D. Schleicher, who had worked on the construction of dynamics in the parameter plane before.

4.3. Edges, Frames, and Piecewise Constructions. For parameters c in the p/q-limb of $\mathcal{M}$, the filled Julia set $\mathcal{K}_c$ has q branches at the fixed point α_c of f_c. A connected subset $\mathcal{E}_c \subset \mathcal{K}_c$ is a *dynamic edge* of order n, if f_c^{n-1} is injective on $\mathcal{E}_c$ and $f_c^{n-1}(\mathcal{E}_c)$ is the part of $\mathcal{K}_c$ between α_c and $-\alpha_c$. (More precisely, f_c^{n-1} shall be injective in a neighborhood of the edge without its vertices.) The edge is characterized by the external angles at the vertices. As c varies, it may still be defined, or it may cease to exist after a bifurcation of preimages of α_c. Now $\mathcal{E}_M \subset \mathcal{M}_{p/q}$ is a *parameter edge*, if for all $c \in \mathcal{E}_M$ the dynamic edge $\mathcal{E}_c$ (with given angles) exists and satisfies $c \in \mathcal{E}_c$, and if $\mathcal{E}_M$ has the same external angles as $\mathcal{E}_c$. In Figures 2–3, $\mathcal{E}_M$ is the parameter edge of order 4 in $\mathcal{M}_{1/3}$.

$\mathcal{M}_{p/q}$ contains a little Mandelbrot set $\mathcal{M}' = c_0 * \mathcal{M}$ of period q (cf. Section 4.4). If a parameter edge $\mathcal{E}_M$ is behind $c_0 * (-1)$, there is a homeomorphism $h : \mathcal{E}_M \to \mathcal{E}_M$ analogous to that of Figures 2–3. Behind the α-type Misiurewicz point $c_0 * (-2)$, edges can be decomposed into subedges and *frames* [8, Section 7]. These frames are constructed recursively, like the intervals in the complement of the middle-third Cantor set. A family of homeomorphisms on subedges shows that all frames on the same edge are mutually homeomorphic, and they form a finer decomposition than the fundamental domains of a single homeomorphism. By permuting the frames (in a monotonous way), new homeomorphisms h are defined piecewise. These may have properties that are not possible when h is constructed from a single surgery. E.g., in contradiction to Section 3.4, h can be constructed such that it is not Lipschitz continuous or not even Hölder continuous at the vertex a of $\mathcal{E}_M$. Or it can map a Misiurewicz point with two external angles to a parameter with irrational angles, which is not a Misiurewicz point.

The notions of edges and frames can be generalized: for parameters c behind the root of any hyperbolic component (except the main cardioid), the filled Julia set $\mathcal{K}_c$ contains two corresponding pre-characteristic points, which take the roles of $\pm\alpha_c$.

4.4. Tuning and Composition of Homeomorphisms. For a center c_0 of period p, there is a "little Mandelbrot set" $\mathcal{M}' \subset \mathcal{M}$ and a tuning map $\mathcal{M} \to \mathcal{M}'$, $x \mapsto y = c_0 * x$ with $0 \mapsto c_0$. Now $\mathcal{K}_y$ contains a "little Julia set" $\mathcal{K}_{y,p}$ around 0, where f_y^p is conjugate to f_x on $\mathcal{K}_x$ [6, 7]. A homeomorphism $h : \mathcal{E}_M \to \mathcal{E}_M$ according to Theorem 1.1 is compatible with tuning in two different ways:

- If $c_0 \in \mathcal{E}_M$, then h is mapping $\mathcal{M}'$ to the little Mandelbrot set at $h(c_0)$: $h(c_0 * x) = (h(c_0)) * x$. Cf. [2].
- For any center $c_0 \in \mathcal{M}$, set $\mathcal{E}'_M := c_0 * \mathcal{E}_M \subset \mathcal{M}$. A new homeomorphism $h' : \mathcal{E}'_M \to \mathcal{E}'_M$ is obtained by composition, i.e., $h'(c_0 * x) := c_0 * (h(x))$. Now $\mathcal{E}'_M$ is obtained by disconnecting $\mathcal{M}$ at a countable family of pinching points, but h' has a natural extension to all of these "decorations" (except for two): the mapping η_c that produced the homeomorphism h is transferred by cutting the little Julia set into strips. The required pinching points do not bifurcate when the parameter y is in a decoration of $\mathcal{E}'_M$, thus the new piecewise construction η'_y works in a whole strip. An example is shown in Figure 5 (left).

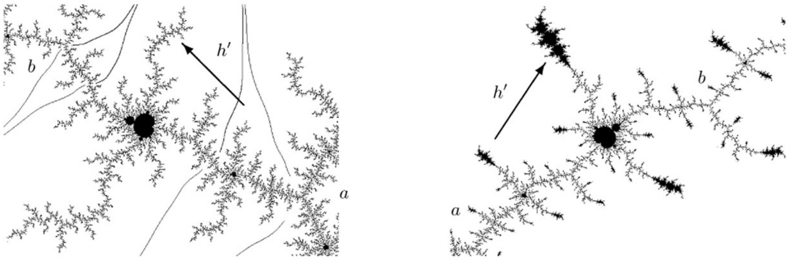


Figure 5: Two homeomorphisms $h' : \mathcal{E}'_M \to \mathcal{E}'_M$ obtained from a similar construction as $h : \mathcal{E}_M \to \mathcal{E}_M$ in Figures 2–3. Left: tuning with the center $c_0 = -1$ yields an edge in the limb $\mathcal{M}_{1/2} \subset \mathcal{M}$. The eight angles $\Theta_i^\pm$ are obtained by tuning those of Figure 2, i.e., replacing the digits 0 by 01 and 1 by 10. h' is defined not only on $c_0 * \mathcal{E}_M$, but on a strip including all decorations. Right: part of the parameter space of cubic polynomials with a persistently indifferent fixed point. The connectedness locus contains copies of a quadratic Siegel Julia set [4]. Again, h' is defined in a strip containing a countable family of decorations attached to the copy of $\mathcal{E}_\mathcal{M}$ as explained in Section 4.5.

The same principle applies, e.g., to crossed renormalization [13], or to the Branner-Douady homeomorphism $\Phi_A : \mathcal{M}_{1/2} \to \mathcal{T} \subset \mathcal{M}_{1/3}$: suppose that $\mathcal{E}_M \subset \mathcal{M}_{1/2}$ and $h : \mathcal{E}_M \to \mathcal{E}_M$ is constructed according to Theorem 1.1, i.e., from a combinatorial $g_c^{(1)}$ according to Definition 3.1. Then $\mathcal{E}'_M := \Phi_A(\mathcal{E}_M)$ is a subset of $\mathcal{M}_{1/3}$, where a countable family of decorations was cut off. Again $h' := \Phi_A \circ h \circ \Phi_A^{-1} : \mathcal{E}'_M \to \mathcal{E}'_M$ extends to a whole strip by transferring the combinatorial construction of $g_c^{(1)}$. If e.g., h is a suitable homeomorphism on the edge from $\gamma_M(5/12)$ to $\gamma_M(11/24)$, then h' is the homeomorphism of Figures 2, 3.

4.5. Other Parameter Spaces. In Theorem 1.1 we obtained homeomorphisms $h : \mathcal{E}_M \to \mathcal{E}_M$ of suitable subsets $\mathcal{E}_M \subset \mathcal{M}$, but the method is not limited to quadratic polynomials. To apply it to other one-dimensional families of polynomials or rational mappings, these mappings must be characterized dynamically. The polynomials of degree d form a $(d-1)$-dimensional family (modulo affine conjugation). Suppose that a one-dimensional subfamily f_c is defined by one or more of the following *critical relations*:

- A critical point of f_c is degenerate, or one critical point is iterated to another one, or critical orbits are related by f_c being even or commuting with a rotation.
- A critical point is preperiodic or periodic (superattracting).
- There is a persistent cycle with multiplier ρ, $0 < |\rho| \le 1$. This cycle is always "catching" one of the critical points, but the choice may change.

An appropriate combination of such relations defines a one-parameter family f_c, where the coefficients and the critical points of f_c are algebraic in c. Locally in the parameter space, there is one *active* or *free* critical point ω_c, whose orbit determines the qualitative dynamics. The other critical points are either linked to ω_c, or their behavior is independent of c. The connectedness locus $\mathcal{M}_f$ contains the parameters c, such that the filled Julia set $\mathcal{K}_c$ of f_c is connected, equivalently $f_c^n(\omega_c) \not\to \infty$, or $\omega_c \in \mathcal{K}_c$. In general ω_c is not defined globally by an analytic function of c, since it may be a multi-valued algebraic function of c, or since a persistent cycle may catch different critical points. But looking at specific families, it will be possible to define Φ_M and parameter rays for suitable subsets of the parameter space, and to understand their landing properties.

Then an analogue of Theorem 1.1 can be proved: for a piecewise defined $g_c^{(1)}$, a quasi-polynomial mapping g_c is constructed analogously to Section 3.2, and straightened to a polynomial. By the critical relations and by normalizing conditions, it will be of the form f_d, and we set $h(c) := d$. (At worst, the normalizing conditions will allow finitely many choices for d.) Note that this procedure will not work, if our family f_c is an arbitrary submanifold of the $(d-1)$-dimensional family of all polynomials, and not defined by critical relations.

When such a theorem is proved, the remaining creative step is the combinatorial definition of $\mathcal{E}_M$ and $g_c^{(1)}$. Some examples can be obtained in the following way: when a non-degenerate critical point is active, the connectedness locus $\mathcal{M}_f$ will contain copies $\mathcal{M}'$ of $\mathcal{M}$ [12]. Starting from a homeomorphism $h : \mathcal{E}_M \to \mathcal{E}_M$, $\mathcal{M}'$ contains a decorated copy of $\mathcal{E}_M$, and the corresponding homeomorphism h' extends to all decorations by an appropriate definition of $g_c^{(1)}$—the angles $\Theta_i^\pm$ are seen at the copy of a quadratic Julia set within $\mathcal{K}_c$, where some iterate of f_c is conjugate to a quadratic polynomial. It remains to check that no other critical orbit is passing through $\overline{V_c \cup W_c}$, then $g_c^{(1)}$ is well-defined. An example is given in Figure 5 (right).

The rational mappings of degree d form a $(2d-2)$-dimensional family (modulo Möbius conjugation). Suppose that a one-dimensional subfamily f_c is defined by critical relations. When there are one or more persistently (super-) attracting cycles, then $\mathcal{K}_c$ shall be the complement of the basin of attraction, and $\mathcal{M}_f$ shall contain those parameters, such that the local free critical point is not attracted. $\partial\mathcal{M}_f$ will be the bifurcation locus [12]. If the persistent cycles are superattracting, we can define dynamic rays and parameter rays by the Boettcher conjugation. When the topology and the landing properties are understood sufficiently well, homeomorphisms can be constructed by quasi-conformal surgery.

An example is provided by cubic Newton methods: f_c has three superattracting fixed points and one free critical point. Parts of the parameter space are shown in Figure 6, cf. [6], [19], [15]. By dynamic rays in the adjacent immediate basins of two fixed points, the Julia set is cut into "strips" to define $g_c^{(1)}$. In both basins, the techniques of Section 3.2 are applied to construct the quasi-Newton mapping g_c. It is straightened to f_d, and a homeomorphism is obtained by $h_0(c) := d$. It is permuting little "almonds," respecting their decomposition into four colors according to the atraction of the free critical point to a root. Similar constructions are possible when one or both of the adjacent components of basins at $\mathcal{E}_c$ are not immediate, i.e., when the hyperbolic components at $\mathcal{E}_M$ are of greater depth [15]. Cf. h_1, h_2 in Figures 6–7.

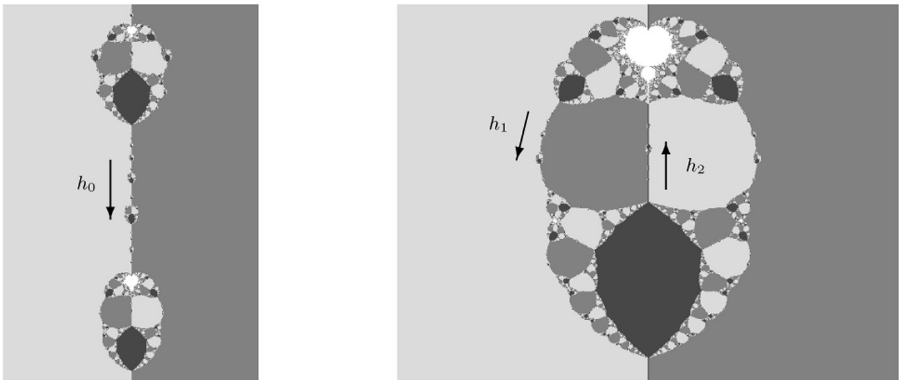


Figure 6: Homeomorphisms in the parameter space of Newton methods for cubic polynomials. Left: an "edge" between the "almonds" of orders 3 and 2. Right: homeomorphisms on edges within the almond of order 2. (The different colors, or shades of gray, indicate that the free critical point is attracted to one of the roots of the corresponding polynomial.)

Cubic Newton methods are understood as matings of cubic polynomials [19], and there are analogous homeomorphisms in the parameter space of cubic polynomials with one superattracting fixed point. Again, the rays used in the piecewise definition of $g_c^{(1)}$ belong to the basins of two attracting fixed points, but one is finite and one at ∞ in the polynomial case. H. Hubbard has suggested to look at quadratic rational mappings with a superattracting cycle, which contain matings of quadratic polynomials. When we try to transfer a known homeomorphism of the Mandelbrot set to this family, in general we will have to use articulated rays to cut the Julia set. Although it may be possible to define $g_c^{(1)}$, it will not be possible to construct the quasi-regular mapping g_c, because the shift discontinuity happens not only within the basin of attraction, but at pinching points of the Julia set as well. For the same reason, it will not be possible to transfer a homeomorphism of $\mathcal{M}$ to a neighborhood of a copy of $\mathcal{M}$ in the cubic Newton family.

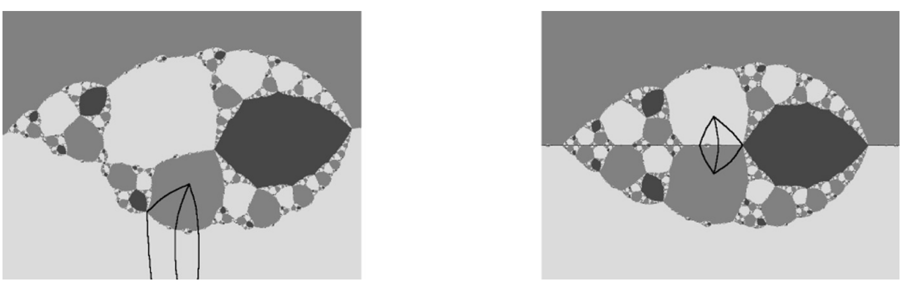


Figure 7: Cutting the Julia set with dynamic rays belonging to two different basins, to define the strips V_c , W_c and the mapping $g_c^{(1)}$. This yields the homeomorphisms h_1 (left) and h_2 (right) in the almond of order 2 (cf. Figure 6).

5. Homeomorphism Groups of $\mathcal{M}$. Denote the group of orientation-preserving homeomorphisms $h : \mathcal{M} \to \mathcal{M}$ by $\mathcal{G}_M$. If two homeomorphisms coincide on $\partial\mathcal{M}$, they encode the same information on the topological structure of $\mathcal{M}$. To exclude these trivial homeomorphisms, some definitions of groups of non-trivial homeomorphisms are suggested as well.

Definition 5.1 (Groups of Homeomorphisms)**.**

1. $\mathcal{G}_M$ is the group of homeomorphisms $h : \mathcal{M} \to \mathcal{M}$ that are orientation-preserving at branch points, and orientation-preserving in the interior of $\mathcal{M}$.
2. $\mathcal{G}_a$ is the group of homeomorphisms $h : \mathcal{M} \to \mathcal{M}$ that are orientation-preserving at branch points, and analytic in the interior of $\mathcal{M}$.
3. $\mathcal{G}_b$ is the group of homeomorphisms $h : \partial\mathcal{M} \to \partial\mathcal{M}$ that are orientation-preserving at branch points, and orientation-preserving on the boundaries of hyperbolic components.
4. $\mathcal{G}_q$ is the factor group $\mathcal{G}_M/\mathcal{G}_1$, where $\mathcal{G}_1$ is the normal subgroup consisting of trivial homeomorphisms: $\mathcal{G}_1 := \{h \in \mathcal{G}_M \mid h = \mathrm{id} \text{ } on \text{ } \partial\mathcal{M}\}$.

$\mathcal{G}_q$ is the most natural definition of non-trivial homeomorphisms. $\mathcal{G}_a$, $\mathcal{G}_b$, $\mathcal{G}_q$ may well turn out to be mutually isomorphic. On $\mathcal{G}_M$, $\mathcal{G}_a$, and $\mathcal{G}_b$, define a metric by

$$\begin{aligned} d(h_1, h_2) &:= ||h_1 - h_2||_\infty + ||h_1^{-1} - h_2^{-1}||_\infty \\ &:= \max |h_1(c) - h_2(c)| + \max |h_1^{-1}(c) - h_2^{-1}(c)|, \end{aligned} \tag{4}$$

where the maxima are taken over $c \in \mathcal{M}$ or $c \in \partial\mathcal{M}$, respectively. $\mathcal{G}_q$ consists of equivalence classes of homeomorphisms coinciding on the boundary, $[h] = h\mathcal{G}_1 = \mathcal{G}_1 h$. Since $\mathcal{G}_1$ is closed, a metric is given by

$$\begin{aligned} d([h_1], [h_2]) := &\inf\{||h_1' - h_2'||_\infty \mid h_1' \in [h_1], h_2' \in [h_2]\} \\ &+ \inf\{||h_1'^{-1} - h_2'^{-1}||_\infty \mid h_1' \in [h_1], h_2' \in [h_2]\} \qquad (5) \\ = &\inf\{||h_1 \circ u - h_2||_\infty + ||h_1^{-1} \circ v - h_2^{-1}||_\infty \mid u, v \in \mathcal{G}_1\}. \qquad (6) \end{aligned}$$

It may be more natural to take the infimum of a sum instead of the sum of infima in (5), i.e., $\inf d(h_1', h_2')$, but I do not know how to prove the triangle inequality in that case. (6) is obtained from (5) by employing the facts that $\mathcal{G}_1$ is normal, and that right translations are isometries of the norm: $||h_1 - h_2||_\infty = ||h_1 \circ h - h_2 \circ h||_\infty$, since $h \in \mathcal{G}_M$ is bijective.

Proposition 5.2 (Topology of Homeomorphisms Groups)**.**

1. $\mathcal{G}_M$, $\mathcal{G}_a$, $\mathcal{G}_b$, $\mathcal{G}_q$ *are complete metric spaces and topological groups, i.e., composition and inversion are continuous.*
2. *For* $\mathcal{G} = \mathcal{G}_M$, $\mathcal{G}_a$, $\mathcal{G}_b$, $\mathcal{G}_q$ *we have: if* Ω_1, Ω_2 *are hyperbolic components, then* $\mathcal{N} := \{h \in \mathcal{G} \mid h(\partial\Omega_1) = \partial\Omega_2\}$ *is open.*

Proof. 1. For $\mathcal{G}_M$, $\mathcal{G}_a$, $\mathcal{G}_b$, the proof is straightforward. But suppose we had used the alternative metric $\widetilde{d}(h_1, h_2) := ||h_1 - h_2||_\infty$, and $(h_n) \subset \mathcal{G}_M$ is a Cauchy sequence in that metric. Then it is converging uniformly to a continuous, surjective $h : \mathcal{E}_M \to \mathcal{E}_M$. If h is injective, then $h_n^{-1} \to h^{-1}$ uniformly. But h need not be injective, a counter-example is constructed in item 2 of [8, Proposition 7.7]. Thus, if we had used $\widetilde{d}$ instead of d, the topology of $\mathcal{G}_M$, $\mathcal{G}_a$, $\mathcal{G}_b$ would be the same, but they would be incomplete metric spaces.

Now suppose $([h_n])$ is a Cauchy sequence in $\mathcal{G}_q$. It is sufficient to show that a subsequence converges, and without restriction we have $d([h_{n+1}],\ [h_n]) \le 3^{-n}$. Choose $u_n,\ v_n \in \mathcal{G}_1$ with

$$||h_{n+1} \circ u_n - h_n||_\infty \le 2^{-n} \quad \text{and} \quad ||h_{n+1}^{-1} \circ v_n - h_n^{-1}||_\infty \le 2^{-n}.$$

Define the sequences

$$\widehat{h}_n := h_n \circ u_{n-1} \circ u_{n-2} \circ \ldots \circ u_1 \quad \text{and} \quad \widetilde{h}_n := h_n^{-1} \circ v_{n-1} \circ v_{n-2} \circ \ldots \circ v_1.$$

Since the maximum norm is invariant under right translations on $\mathcal{M}$, they satisfy

$$||\widehat{h}_{n+1} - \widehat{h}_n||_\infty \le 2^{-n} \quad \text{and} \quad ||\widetilde{h}_{n+1} - \widetilde{h}_n||_\infty \le 2^{-n},$$

and there are continuous functions $\widehat{h}$, $\widetilde{h}$ with $\widehat{h}_n \to \widehat{h}$ and $\widetilde{h}_n \to \widetilde{h}$ uniformly. Now $\widehat{h} \circ \widetilde{h}$ and $\widetilde{h} \circ \widehat{h}$ are uniform limits of a sequence in $\mathcal{G}_1$, thus surjective, and $\widehat{h}$ is a homeomorphism. We have $\widehat{h}_n \to \widehat{h}$ in $\mathcal{G}_M$ and $[h_n] = [\widehat{h}_n] \to [\widehat{h}]$ in $\mathcal{G}_q$, therefore $\mathcal{G}_q$ is complete.

2. Hyperbolic components can be distinguished topologically from non-hyperbolic components, since only the boundary of a hyperbolic component contains a countable dense set of pinching points (by the Branch Theorem [16]). Thus every homeomorphism of $\mathcal{M}$ or $\partial\mathcal{M}$ is permuting the set of hyperbolic components or of their boundaries, respectively. Fix $a,\ b \in \partial\Omega_2$, and choose $\varepsilon > 0$ such that no hyperbolic component $\neq \Omega_2$ is meeting both of the disks of radius ε around a and b. This is possible, since there are several external rays landing at $\partial\Omega_2$. If $h_0 \in \mathcal{N}$ and $h \in \mathcal{G}$ with $d(h,\ h_0) < \varepsilon$, then $|h(h_0^{-1}(a)) - a| < \varepsilon$ and $|h(h_0^{-1}(b)) - b| < \varepsilon$, thus $h(\partial\Omega_1) = \partial\Omega_2$. (Analogously for the classes in $\mathcal{G}_q$.) □

Theorem 5.3 (Groups of Non-Trivial Homeomorphisms)**.** *The groups of non-trivial homeomorphisms of $\mathcal{M}$ or $\partial\mathcal{M}$—$\mathcal{G}_a$, $\mathcal{G}_b$, and $\mathcal{G}_q$—share the following properties:*

1. *They have the cardinality of the continuum $\mathbb{R}$, and they are totally disconnected.*
2. *They are perfect, and not compact* (*not even locally compact*)*.*
3. *A family of homeomorphisms $\mathcal{F} \subset \mathcal{G}_a$, $\mathcal{G}_b$, $\mathcal{G}_q$ is called normal, if its closure is sequentially compact. A necessary condition is that for every hyperbolic component $\Omega \subset \mathcal{M}$, there are only finitely many components of the form $h^{\pm 1}(\Omega)$, $h \in \mathcal{F}$. If $\mathcal{M}$ is locally connected, this condition will be sufficient for $\mathcal{F}$ being normal.*

By composition, the homeomorphisms constructed by surgery according to Theorem 1.1 generate a countable subgroup of $\mathcal{G}_a$, $\mathcal{G}_b$ or $\mathcal{G}_q$. Will it be dense?—For $\mathcal{G}_M$, items 1 and 3 are wrong, and item 2 is true but trivial. Hence the motivation to consider the groups of non-trivial homeomorphisms. The same results hold for the analogous groups, where the condition of preserving the orientation is dropped.

Proof. We prove the statements for $\mathcal{G}_a$, the case of $\mathcal{G}_b$ or $\mathcal{G}_q$ is similar. There is a sequence of disjoint subsets $\mathcal{E}_n \subset \mathcal{M}$ with $\operatorname{diam}(\mathcal{E}_n) \to 0$, and a sequence of analytic homeomorphisms $h_n : \mathcal{M} \to \mathcal{M}$, such that $h_n = \mathrm{id}$ on $\mathcal{M} \backslash \mathcal{E}_n$, $h_n \neq \mathrm{id}$. To construct these, fix a homeomorphism $h_* : \mathcal{E}_M \to \mathcal{E}_M$ according to Theorem 1.1, e.g., that of Figures 2 and 3. Choose $\mathcal{E}_0 \subset \mathcal{E}_M$ and a homeomorphism $h_0 : \mathcal{E}_0 \to \mathcal{E}_0$, $h_0 \neq \mathrm{id}$, such that $\mathcal{E}_0$ is contained in a fundamental domain of h_*. This is possible e.g., by the tuning construction from Section 4.4. Then set $h_n := h_*^n \circ h_0 \circ h_*^{-n}$ on $\mathcal{E}_n := h_*^n(\mathcal{E}_0)$, and extend it by the identity to a homeomorphism of $\mathcal{M}$. We have diam $(\mathcal{E}_n) \to 0$ by the scaling properties of $\mathcal{M}$ at Misiurewicz points [18]. —An alternative approach is as follows: construct homeomorphisms $h_n : \mathcal{E}_n \to \mathcal{E}_n$, such that $\mathcal{E}_n$ is contained in the limb $\mathcal{M}_{1/n}$, then $\operatorname{diam}(\mathcal{E}_n) \to 0$ by the Yoccoz inequality. These homeomorphisms can be constructed by tuning, or at β-type Misiurewicz points according to Section 4.2, or on edges (Section 4.3). All of the homeomorphisms constructed below extend to homeomorphisms of $\mathbb{C}$, cf. item 3 of Remark 3.2. (If $\mathcal{M}$ is locally connected, all homeomorphisms in $\mathcal{G}_M$, $\mathcal{G}_a$, or $\mathcal{G}_b$ will extend to homeomorphisms of $\mathbb{C}$.)

1. We construct an injection $(0,\ 1) \to \mathcal{G}_a,\ x \mapsto h$ as follows: expand x in binary digits (not ending on $\overline{1}$). Set $h := h_n$ or $h := \mathrm{id}$ on $\mathcal{E}_n$, if the n-th digit is 1 or 0, respectively, and $h := \mathrm{id}$ on $\mathcal{M} \backslash \bigcup \mathcal{E}_n$. Although the sequence of sets $\mathcal{E}_n$ will accumulate somewhere, continuity of h can be shown by employing diam $(\mathcal{E}_n) \to 0$. —Conversely, to obtain an injection $\mathcal{G}_a \to (0,\ 1),\ h \mapsto x$, enumerate the hyperbolic components $(\Omega_n)_{n \in \mathbb{N}}$, and denote the n-th prime number by p_n. Now x shall have the digit 1 at the place p_n^m, iff $h : \Omega_n \to \Omega_m$. The mapping $h \mapsto x$ is injective, since the group homomorphism from $\mathcal{G}_a$ to the permutation group of hyperbolic components is injective: if h is mapping every hyperbolic component to itself, it is fixing the points of intersection of closures of hyperbolic components, i.e., all roots of satellite components. These are dense in $\partial\mathcal{M}$, thus $h = \mathrm{id}$. —By the two injections, $|\mathcal{G}_a| = |(0,\ 1)| = |\mathbb{R}|$.

If $h_1,\ h_2 \in \mathcal{G}_a$ with $h_1 \neq h_2$, there is a hyperbolic component Ω with $h_1(\Omega) \neq h_2(\Omega)$. By Proposition 5.2, $\mathcal{N} := \{h \in \mathcal{G}_a \mid h(\Omega) = h_1(\Omega)\}$ is an open neighborhood of h_1, and $\mathcal{G}_a \backslash \mathcal{N} = \bigcup \{h \in \mathcal{G}_a \mid h(\Omega) = \Omega'\}$ is an open neighborhood of h_2, where the union is taken over all hyperbolic components $\Omega' \neq h_1(\Omega)$. Thus h_1 and h_2 belong to different connected components, and $\mathcal{G}_a$ is totally disconnected.

2. We have $d(h_n,\ \mathrm{id}) \leq 2 \operatorname{diam}(\mathcal{E}_n) \to 0$ as $n \to \infty$, thus id is not isolated in $\mathcal{G}_a$. Since composition is continuous, no point is isolated, and $\mathcal{G}_a$ is perfect.

Choose a homeomorphism $h : \mathcal{E}_M \to \mathcal{E}_M$ according to Theorem 1.1, which is expanding at a and contracting at b, extend it by the identity to $h \in \mathcal{G}_a$. The iterates of h satisfy $h^k(a) = a$ and $h^k(c) \to b$ for all $c \in \mathcal{E}_M \backslash \{a\}$, thus the pointwise limit of $(h^k)_{k \in \mathbb{N}}$ is not continuous. The sequence does not contain a subsequence converging uniformly, and $\mathcal{G}_a$ is not sequentially compact, a fortiori not compact. —If $\mathcal{N}$ is a neighborhood of id in $\mathcal{G}_a$, fix an n such that $\mathcal{N}$ contains the ball of radius $2\,\mathrm{diam}(\mathcal{E}_n)$ around id, then $\mathcal{N}$ contains the sequence $(h_n^k)_{k\in\mathbb{N}}$. Thus $\mathcal{N}$ is not compact, and $\mathcal{G}_a$ is not locally compact.

3. When $\mathcal{F}$ does not satisfy the finiteness condition, there is a sequence $(h_n)\subset\mathcal{F}$ and a hyperbolic component Ω, such that the period of $h_n(\Omega)$ (or $h_n^{-1}(\Omega)$) diverges. Assume $h_n\to h$, then $h_n(\Omega)=h(\Omega)$ for $n\geq N_0$ according to Proposition 5.2, a contradiction. If $\mathcal{F}$ satisfies the finiteness condition, a diagonal procedure yields a subsequence which is eventually constant on every hyperbolic component, thus respecting the partial order of hyperbolic components. Assuming local connectivity, all fibers are trivial [16], and $\lim h_n$ is obtained analogously to [8, Section 9.3]. □

Two rational angles with odd denominators are *Lavaurs-equivalent*, if the corresponding parameter rays are landing at the same root. Denote the closure of this equivalence relation on S^1 by $\sim$. The *abstract Mandelbrot set* is the quotient space $S^1/\!\sim$ [10], it is a combinatorial model for $\partial\mathcal{M}$, which will be homeomorphic to $\partial\mathcal{M}$ if $\mathcal{M}$ is locally connected. (It is analogous to Douady's *pinched disk model* of $\mathcal{M}$.) Orientation-preserving homeomorphisms of the abstract Mandelbrot set are described by orientation-preserving homeomorphisms $\mathbf{H}: S^1\to S^1$ that are compatible with $\sim$. According to Section 4.1, every homeomorphism $h:\mathcal{E}_M\to\mathcal{E}_M$ constructed by surgery defines such a circle homeomorphism (extended by the identity), and the homeomorphism group of $S^1/\!\sim$ has the properties given in Theorem 5.3. In fact, these homeomorphisms of the abstract Mandelbrot set can be constructed in a purely combinatorial way, without using quasi-conformal surgery [8].

Acknowledgment. Many people have contributed to this work by inspiring discussions and helpful suggestions. I wish to thank in particular Mohamed Barakat, Walter Bergweiler, Volker Enss, Nûria Fagella, John Hamal Hubbard, Gerhard Jank, Karsten Keller, Hartje Kriete, Fernando Lledô, Olaf Post, Johannes Riedl, Pascale Roesch, and Dierk Schleicher.

I am especially happy to contribute this paper to the proceedings of a conference in honor of Bodil Branner, since I have learned surgery from her papers [1], [2].

References

[1] B. Branner, A. Douady, Surgery on complex polynomials, in: *Holomorphic dynamics*, Proc. 2nd Int. Colloq. Dyn. Syst., Mexico City, LNM **1345**, 11–72 (1988).

[2] B. Branner, N. Fagella, Homeomorphisms between limbs of the Mandelbrot set, J. Geom. Anal. **9**, 327–390 (1999).

[3] B. Branner, N. Fagella, Extensions of Homeomorphisms between Limbs of the Mandelbrot Set, Conform. Geom. Dyn. **5**, 100–139 (2001).

[4] X. Buff and C. Henriksen, Julia sets in parameter spaces, Comm. Math. Phys. **220**, 333–375 (2001).

[5] L. Carleson and T. W. Gamelin, *Complex dynamics*, Springer, New York, 1993.

[6] A. Douady, J. H. Hubbard, On the dynamics of polynomial-like mappings, Ann. Sci. École Norm. Sup. **18**, 287–343 (1985).

[7] P. Haïssinsky, Modulation dans l'ensemble de Mandelbrot, in: *The Mandelbrot Set, Theme and Variations*, Tan L. ed., LMS Lecture Notes **274**, Cambridge University Press 2000.

[8] W. JUNG, *Homeomorphisms on Edges of the Mandelbrot Set*, Ph.D. thesis RWTH Aachen 2002. Available from www.bth.rwth-aachen.de/opus/ and www.math.sunysb.edu/dynamics/.

[9] W. JUNG, Self-Similarity and Homeomorphisms of the Mandelbrot Set, in preparation.

[10] K. KELLER, *Invariant Factors, Julia Equivalences and the* (*Abstract*) *Mandelbrot Set*, LNM **1732**, Springer 2000.

[11] O. LEHTO, K. I. VIRTANEN, *Quasiconformal Mappings in the Plane*, Springer 1973.

[12] C. T. MCMULLEN, The Mandelbrot set is universal, in: *The Mandelbrot Set, Theme and Variations*, Tan L. ed., LMS Lecture Notes **274**, Cambridge University Press 2000.

[13] J. RIEDL and D. SCHLEICHER, On Crossed Renormalization of Quadratic Polynomials, in: *Proceedings of the* 1997 *conference on holomorphic dynamics*, RIMS Kokyuroku **1042**, 11–31, Kyoto 1998.

[14] J. RIEDL, *Arcs in Multibrot Sets, Locally Connected Julia Sets and Their Construction by Quasiconformal Surgery*, Ph.D. thesis TU München 2000.

[15] P. ROESCH, *Topologie locale des methodés de Newton cubiques*, Ph.D. thesis ENS de Lyon 1997.

[16] D. SCHLEICHER, On Fibers and Local Connectivity of Mandelbrot and Multibrot Sets, IMS-preprint 98–13a (1998).

[17] D. SCHLEICHER, Rational Parameter Rays of the Mandelbrot Set, Astérisque **261**, 405–443 (2000).

[18] L. TAN, Similarity between the Mandelbrot set and Julia sets, Comm. Math. Phys. 134, 587–617 (1990).

[19] L. TAN, Branched coverings and cubic Newton maps, Fundam. Math. **154**, 207–260 (1997).

Wolf jung Stud.Ref. am Studienseminar Mönchengladbach
Abteistrasse 43–45,
41061 Mönchengladbach,
Germany.
jung@iram.rwth-aachen.de
http://www.iram.rwth-aachen.de/∼jung

Arnold Disks and the Moduli of Herman Rings of the Complex Standard Family

Núria Fagella[*]
Christian Henriksen[†]

1st August 2005

Dedicated to Bodil Branner on her 60th birthday

Abstract. We consider the Arnold family of analytic diffeomorphisms of the circle $x \mapsto x + t + \frac{a}{2\pi}\sin(2\pi x)$ mod (1), where $a, t \in [0, 1)$ and its complexification $f_{\lambda,a}(z) = \lambda z e^{\frac{a}{2}(z-\frac{1}{z})}$, with $\lambda = e^{2\pi i t}$ a holomorphic self map of $\mathbb{C}^*$. The parameter space contains the well known Arnold tongues $\mathcal{T}_\alpha$ for $\alpha \in [0, 1)$ being the rotation number. We are interested in the parameters that belong to the irrational tongues and in particular in those for which the map has a Herman ring. Our goal in this paper is twofold. First we are interested in studying how the modulus of this Herman ring varies in terms of the parameter a, when a tends to 0 along the curve $\mathcal{T}_\alpha$. We survey the different results that describe this variation including the complexification of part of the Arnold tongues (called *Arnold disks*) which leads to the best estimate. To work with this complex parameter values we use the concept of the *twist coordinate*, a measure of how far from symmetric the Herman rings are. Our second goal is to investigate the slice of parameter space that contains all maps in the family with twist coordinate equal to one half, proving for example that this is a plane in $\mathbb{C}^2$. We show a computer picture of this slice of parameter space and we also present some numerical algorithms that allow us to compute new drawings of non–symmetric Herman rings of various moduli.

1. Introduction. In this paper we deal with the holomorphic maps of $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ given by

$$f_{\lambda,a}(z) = \lambda z e^{\frac{a}{2}(z-\frac{1}{z})},$$

for $\lambda = e^{2\pi i t} \in \mathbb{S}^1$ and $a \in [0, 1)$ (to start with). This family is called *the complex Arnold* (*or standard*) *family*, since $f_{\lambda,a}$ restricted to the unit circle, corresponds with the well known Arnold family of circle maps

$$x \mapsto x + t + \frac{a}{2\pi}\sin(2\pi x) \quad (\text{mod } 1).$$

For the given range of parameter values, the maps $f_{\lambda,a}$ are symmetric with respect to the unit circle, and they have two critical points which lie on the negative real line. The points at 0 and ∞ are essential singularities. Since the restriction of these maps to the unit circle is a diffeomorphism of the circle, we may assign a well defined rotation number to each

[*] Partially supported by CIRIT grant 2001SGR-70, MCyT grants BFM2002-01344 and BFM2003-9504.
[†] Supported by SNF Steno fellowship.

member of the family. In this paper we consider the maps with irrational rotation numbers. (See [F] for a description of the dynamics for rational values of the rotation number.)

We consider the level sets of a given rotation number in the (t,a)–parameter plane. Given $\alpha \in [0, 1)$ the set $\mathcal{T}_\alpha = \{(t,a) \in [0, 1) \times [0, 1) \mid \mathrm{rot}\#(f_{\lambda,a}) = \alpha,\ \lambda = e^{2\pi i t}\}$ is called the Arnold tongue of rotation number α. It is well known that $\mathcal{T}_\alpha$ is a set with interior if $\alpha \in \mathbb{Q}$ and, if $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ then $\mathcal{T}_\alpha$ is a Lipschitz curve connecting $(\alpha, 0)$ with $(t', 1)$ for some $t' \in (0, 1)$ [A]. Indeed, the curve can be parametrized as $\{(t(a),a) \mid 0 \le t \le 1\}$ where the function $a \mapsto t(a)$ is Lipschitz. See Figure 1.

Let α be the rotation number of $f_{\lambda,a}$. It follows from theorems of Poincaré and Denjoy (see e.g. [1]) that, if α is irrational, then $f_{\lambda,a}$ is topologically conjugate to the the rigid rotation $\mathcal{R}_\alpha(z) = e^{2\pi i \alpha} z$. This means that there exists a homemorphism $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ such that $f_{\lambda,a} \circ \phi = \phi \circ \mathcal{R}_\alpha$ on the unit circle. If ϕ can be chosen to be real analytic, we say that $f_{\lambda,a}$ is *analytically linearizable*.

If a map can be analytically linearized on the unit circle, then the conjugacy ϕ extends (also as a conjugacy) to a neighborhood of the unit circle. As a consequence, there exists a maximal domain, H, called a *Herman ring* around the unit circle where the map can be linearized. That is, there exist a number $0 < r < 1$ and a conformal map $\phi : A_r \to H$ which conjugates $\mathcal{R}_\alpha$ to $f_{\lambda,a}$, where $A_r = \{z \in \mathbb{C} \mid r < |z| < 1\}$. See Figure 2.

The *modulus of* H is $m = \mathrm{mod}(H) = \frac{1}{2\pi} \log \frac{1}{r}$ and we define the *size of* H to be $s = \mathrm{size}(H) = e^{\pi(H)}$. Observe that A_r is conformally equivalent to an annulus of the form $\{\frac{1}{s} < |z| < s\}$.

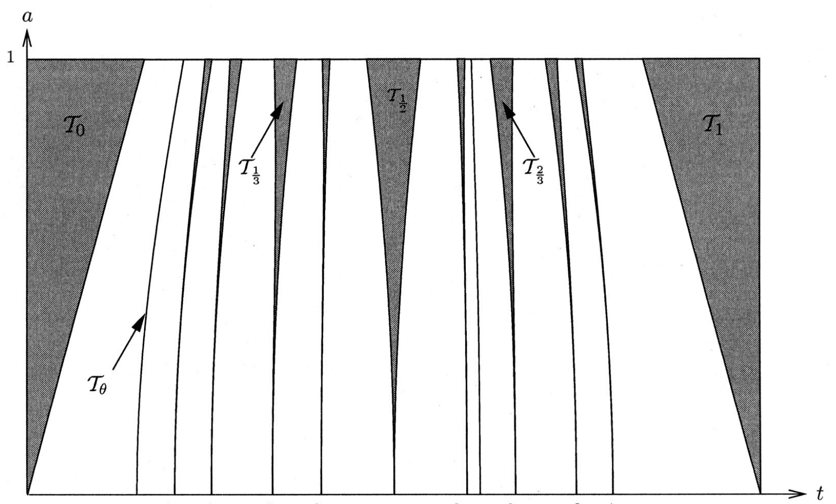


Figure 1: Rational Arnold tongues in the parameter space of the family $f_{\lambda,a}$ for $\lambda = e^{2\pi i t}$, $t \in \mathbb{R}/\mathbb{Z}$, up to denominator 5. Irrational tongues for $\gamma = \frac{\sqrt{5}-1}{2}$ and $\theta = \sqrt[5]{2} - 1$. (Picture made by Lukas Geyer.)

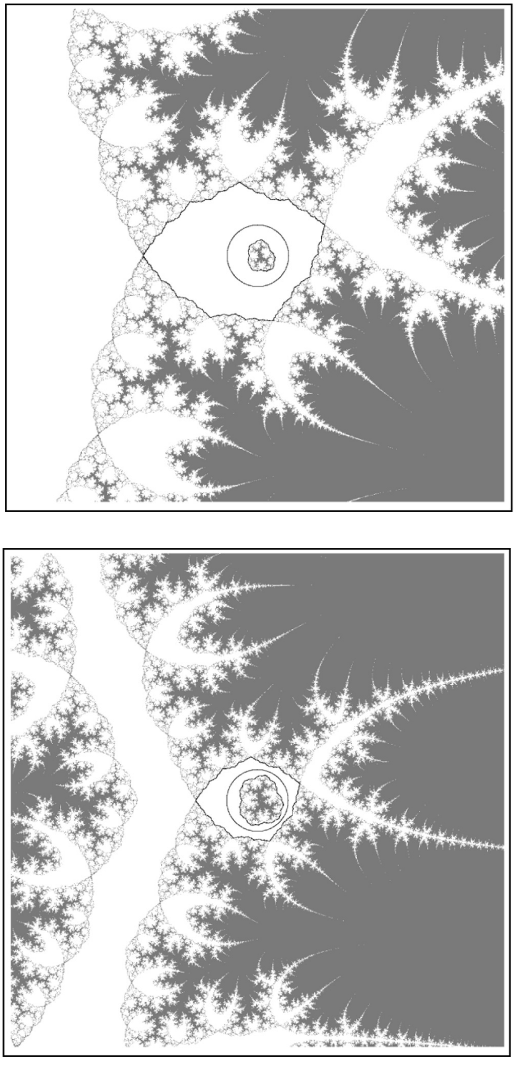


Figure 2: Herman rings for $f_{\lambda, a}$ where we have chosen the parameters $\lambda \in \mathbb{S}^1$ and $a \in (0, 1)$ so that the rotation number equals the golden mean. The unit circle is drawn inside each of the rings. Range: $[-8, 8] \times [-8, 8]$.

A natural question, not yet solved for general functions, is to know which (optimal) conditions on the map and the rotation number allow us to conclude that the map is analytically linearizable. Works of Rüssmann [Rü], Herman [Her] and Yoccoz [Y] conclude that an analytic circle map sufficiently close to a rigid rotation and whose rotation number is a Brjuno number, is always analytically linearizable. In our case, the condition on the map translates into requiring that the parameter a be small enough. On the other hand, for this particular family, it is known that the Brjuno condition is optimal in the following sense: any member of the Arnold family which is analytically linearizable must have a Brjuno rotation number. This was proven by Geyer in [G], using holomorphic surgery to relate the complex Arnold family to the semistandard map $E_\alpha(z) = e^{2\pi i\alpha} z e^z$, and then establishing the optimality of the Brjuno condition for the maps E_α (see Proposition 2.1).

The semistandard family E_α is in many ways very related to the complex Arnold family. It is often fruitful to rescale the Arnold family to make it a perturbation not of the rigid rotation but of the semistandard map. Indeed, if we change variables by letting $w = \frac{az}{2}$ we obtain a rescaled family

$$g_{\lambda,a}(w) = \lambda w e^w e^{-\frac{a^2}{4w}}.$$

Observe that the invariant circle is now that of radius $a/2$. When $a = 0$, the singular limit of this family is the semistandard map. It is often very convenient to work with the rescaled Arnold family, and in fact we shall do so in many parts of the paper. Since both families are conjugate to each other, the linearizability problems are equivalent.

Observe that for all $\alpha \in \mathbb{C}$, the maps $E_\alpha(z)$ are entire transcendental maps which have $z = 0$ as a fixed point of derivative $\lambda = e^{2\pi i\alpha}$. Hence for $\alpha \in \mathbb{R}/\mathbb{Z}$ this is a neutral fixed point. The linearizability problem for fixed points is very related to the one for circle maps. As before, it consists of knowing under which conditions the map is conformally conjugate to the linear map $z \mapsto \lambda z$, although in this case we require the conjugacy to hold in a neighborhood of the fixed point. When the fixed point is linearizable, the maximal neighborhood Δ_α where this is possible is called a *Siegel disk* (See Figure 3). Hence, if E_α has a Siegel disk Δ_α around 0, there exists a conformal map $\phi : \mathbb{D} \to \Delta_\alpha$ mapping 0 to 0, such that $E_\alpha(\phi(z)) = \phi(\lambda z)$. The quantity $r_\alpha = |\phi'(0)|$ is called *the conformal radius* of Δ_α.

The linearizability problem for the semistandard map is completely solved, in the sense that it is known that E_α is linearizable around $z = 0$ if and only if α is a Brjuno number ([Bru, G]).

We now return to the Arnold family. Fix a Brjuno number α. We consider the parameter values for which the rotation number of $f_{\lambda,a}$ is α, and the map is analytically linearizable. That is, the piece (or pieces, a priori) of the Arnold tongue $\mathcal{T}_\alpha$ for which we find a Herman ring in the dynamical plane of $f_{\lambda,a}$. We are interested in understanding how the modulus or the size of the Herman ring varies in terms of the parameter a, precisely when a tends to zero. With this goal in mind, we present a survey of the results that lead to these type of estimates. We do this in two parts: one looking at the "real" parameter space (Section 2) and two, considering its complexification (Section 3), i.e., allowing λ and a to be complex and studying the complex version of (the linearizable part of) the Arnold tongues, called *Arnold disks*. We see how this last point of view leads to the best estimate on the variation of the modulus which is the following.

Theorem 1. *Let α be a fixed Brjuno number and consider the Arnold tongue $\mathcal{T}_\alpha$ of rotation number α. Let $(\lambda(a), a) \in \mathcal{T}_\alpha$ and a be small enough so that $f_{\lambda(a),a}$ has*

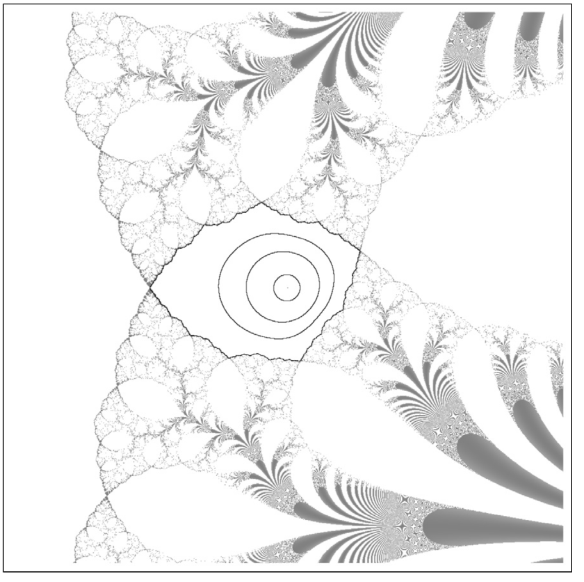


Figure 3: Siegel disk of the function $E_\alpha(z) = e^{2\pi i\alpha} z e^z$, with rotation number α, equal to the golden mean. Some orbits have been drawn inside the Siegel disk. Range: $[-2, 2] \times [-2, 2]$.

a Herman ring. Let $m(a)$ be its modulus and $s(a)$ the corresponding size. Then, as $a \to 0$,

$$s(a) = e^{\pi m(a)} = \frac{2r_\alpha}{a} + \mathcal{O}(a),$$

where r_α is the conformal radius of the Siegel disk of the semistandard map E_α.

If we work with the rescaled Arnold family, the moduli of the rings are obviously the same. But changing variables also in the conjugation plane, we see that the scaled Herman ring is conformally equivalent to an annulus of the form $\{\frac{a^2}{4\tilde{s}(a)} < |z| < \tilde{s}(a)\}$ where $\tilde{s}(a) := \frac{a}{2} s(a)$. Observe that this annulus has the circle of radius $a/2$ as the equator, exactly as the ring does. The quantity $\tilde{s}(a)$ is not a conformal invariant.

Using this terminology, Theorem 1 for the rescaled Arnold family reads as follows.

Theorem 2. *Let α be a fixed Brjuno number and consider the Arnold tongue $\mathcal{T}_\alpha$ of rotation number α. Let $(\lambda(a), a) \in \mathcal{T}_\alpha$ and a be small enough so that $f_{\lambda(a),a}$ and hence $g_{\lambda(a),a}$ have a Herman ring whose modulus is $m(a)$ and whose size is $s(a)$. Then, as $a \to 0$, the quantity $\tilde{s}(a)$ has a limit. More precisely,*

$$\tilde{s}(a) = r_\alpha + \mathcal{O}(a^2),$$

where r_α is the conformal radius of the Siegel disk of the semistandard map E_α.

Intuitively, one can say that the limit when $a \to 0$, of the Herman rings of rotation number α of the rescaled Arnold family are the Siegel disk of the semistandard map E_α.

The second part of the paper (see Section 4) is devoted to study a particular slice of the complex parameter space, more precisely the slice containing those maps whose Herman rings have their boundaries rotated half a turn with respect to each other. We first describe the location of this slice in $\mathbb{C}^2$ (see Theorem 4.1) and show a computer drawing of it.

Finally, Section 5 is dedicated to numerics. The computer drawings in this paper needed some new algorithms to be developed, given the difficulties that one encounters when the symmetries of the map are no longer present. In this final part we present these algorithms which are reusable for other types of functions.

2. Real parameter space. In this section we present two results. The first one concerns the parametrization of the linearizable piece of an irrational Arnold tongue and it is the "real" version of Theorem 5 in Section 3. The second result is a first estimate of the size of Herman rings in terms of the parameter a which was obtained in [FSV]. Given a Brjuno number α and its Arnold tongue $\mathcal{T}_\alpha$, we define $\mathcal{T}_\alpha^{\mathrm{AL}}$ as the analytically linearizable part of $\mathcal{T}_\alpha$, i.e., the set of parameter values $(\lambda, a) \in \mathcal{T}_\alpha$ such that $f_{\lambda,a}$ has a Herman ring around $\mathbb{S}^1$.

Theorem 3 ([FG]). *Fix α a Brjuno number and let $f_{\lambda,a}(z) = \lambda z e^{\frac{a}{2}(z-\frac{1}{z})}$. Then, there exists an $\mathbb{R}-$analytic parametrization*

$$\begin{aligned} \mathcal{F}_\alpha : (0,1) &\longrightarrow \mathcal{T}_\alpha^{\mathrm{AL}} \\ \delta &\longmapsto \mathcal{F}_\alpha(\delta) = (\lambda(\delta), a(\delta)) \end{aligned}$$

such that:

(a) *for all $\delta \in (0,1)$, the* map $f_{\lambda(\delta),a(\delta)}$ *has a Herman ring of modulus $m(\delta) = \frac{1}{\pi}\log\frac{1}{\delta}$ and rotation number α;*
(b) *$\delta \mapsto a(\delta)$ is strictly increasing;*
(c) *$a(\delta) \to 0$ when $\delta \to 0$ and $\lim_{\delta \to 1} a(\delta) = a_* \leq 1$;*
(d) *for all $(\lambda, a) \in \mathcal{T}_\alpha$ such that $a \geq a_*$, the* map $f_{\lambda,a}$ *has no Herman ring.*

This theorem describes the sets $\mathcal{T}_\alpha^{\mathrm{AL}}$ as connected $\mathbb{R}-$analytic curves that might be the entire Arnold tongue $\mathcal{T}_\alpha$. Moreover, it gives the precise modulus of the Herman ring for each of the parameters δ. On one hand as δ tends to 0, the parameter a tends to 0 and the modulus of the ring tends to infinity (we consider the rigid rotation as having a degenerate Herman ring of infinite modulus). On the other hand, as δ tends to 1, the Herman ring gets thinner and thinner, having in the limit a degenerate Herman ring which contains the unit circle.

Theorem 3 is proven by quasiconformal surgery. We give here an idea of its proof since it illustrates quite well the complex case of the next section.

Proof. Since α is a Brjuno number, for a small enough the map $f_{\lambda,a}$ has a Herman ring. Let us fix a base point, i.e., a pair of parameters (λ_1, a_1) in the Arnold tongue $\mathcal{T}_\alpha$, such that $f_1 := f_{\lambda_1,a_1}$ has a Herman ring H_1 whose modulus we denote by m_1.

Now, given any $s \in (0,\infty)$ the goal is to construct a new map $f_{\lambda(s),a(s)}$ with a Herman ring H_s of modulus $m(s) = sm_1$. Moreover, we want to do this construction in such a way that the map $s \mapsto (\lambda(s),a(s))$ is real analytic and has all the required properties (like monotonicity of $a(s)$). Once this is proven, it is not hard to see that the curve can be reparametrized as desired not depending on a base point.

With this goal in mind, we make a surgery construction which consists only of changing the complex structure of the original map. If H_1 is the Herman ring of f_1, it means that there exists a conformal map $\phi_1 : A_r \to H_1$ where $r = e^{-2\pi m_1}$, which conjugates $\mathcal{R}_\alpha$ to f_1. We now compose this map with a quasiconformal map $\varphi_s : A_r \to A_{r^s}$ which maps circles to circles. In particular we want it to leave the unit circle invariant and to send the circle of radius r to the circle of radius r^s. Such a map is not hard to compute explicitly, especially if we do so in the covering space of the annulus. It is easy to check that φ_s conjugates $\mathcal{R}_\alpha$ to itself.

We now proceed to change the complex structure on the dynamical plane of f_1. We first change it on the ring H_1 by pulling back the standard complex structure σ_0 on A_{r^s} by the map $\varphi_s \circ \phi_1^{-1}$. This defines a complex structure σ_s on H_1 which has bounded distortion (it is a pull-back by a quasiconformal map) and is invariant under f_1. We then extend σ_s to the whole dynamical plane by using the dynamics of f_1, i.e., pulling back by f_1^n to all the n–th preimages of H_1, and seting $\sigma_s = \sigma_0$ at every point that never falls on H_1 under iteration. This process defines an f_1–invariant complex structure σ_s in all of $\mathbb{C}^*$ with bounded dilatation. By the Measurable Riemann Mapping Theorem, this structure can be integrated, i.e., there exists a quasiconformal homeomorphism $\psi_s : \mathbb{C} \to \mathbb{C}$ that transports σ_s to σ_0. Hence, the map $f_s := \psi_s \circ f_1 \circ \psi_s^{-1}$ is holomorphic and quasiconformally conjugate to f_1. The following diagram commutes.

$$\begin{array}{ccc} H_s & \xrightarrow{f_s} & H_s \\ \phi_s\uparrow & & \uparrow\phi_s \\ H_1 & \xrightarrow{f_1} & H_1 \\ \phi_1\uparrow & & \uparrow\phi_1 \\ A_r & \xrightarrow{\mathcal{R}_\alpha} & A_r \\ \varphi_s\uparrow & & \uparrow\varphi_s \\ A_{r^s} & \xrightarrow{\mathcal{R}_\alpha} & A_{r^s} \end{array}$$

The set $H_s = \psi_s(H_1)$ is a Herman ring for f_s since the composition $\varphi_s \circ \phi_1^{-1} \circ \psi_s^{-1} : H_s \to A_{r^s}$ is a conformal conjugacy between f_s and $\mathcal{R}_\alpha$. From here one can also see that the modulus of the new ring H_s is equal to $\frac{1}{2\pi}\log\frac{1}{r^s} = sm_1$. Furthermore, f_s must be a member of the complex Arnold family and therefore $f_s = f_{\lambda(s),a(s)}$. This defines the map $s \mapsto (\lambda(s),a(s))$ with the required properties. $\square$

In view of the theorem above one can ask exactly how the modulus of the Herman ring is tending to infinity, as the parameter a tends to 0. As a first estimate we have the following result, which connects the size of the Herman rings with the conformal radius of the semistandard map of the same rotation number.

Theorem 4 ([FSV]). *Let α be a Brjuno number and r_α the conformal radius of the Siegel disk of the semistandard map $E_\alpha(z) = e^{2\pi i\alpha} z e^z$. Let $s(a)$ and $m(a)$ be the size and the modulus of the Herman ring of $f_{\lambda(a),a}$ respectively, with $(\lambda(a),a) \in \mathcal{T}_\alpha^{\mathrm{AL}}$. Then,*

$$s(a) = e^{\pi m(a)} = \frac{2}{a}(r_\alpha + \mathcal{O}(a \log a)).$$

The proof of Theorem 4 relies on understanding how the maps of the Arnold family are related to the semistandard map E_α. We saw in the introduction how one can relate them by means of a rescaling depending on a, but to really study the limit, it is better to perform a surgery construction that shows why these two families of maps are related. The construction is originally due to Shishikura [3] who used it to construct examples of rational maps with Herman rings starting from polynomials with Siegel disks (and vice-versa). Later on, Geyer [G] adapted the proof to the Arnold family and the semistandard map. The result of the construction is summarized in the following proposition.

Proposition 2.1. *Suppose $f = f_{\lambda,a}$ has a fixed Herman ring H with rotation number α. Then the semistandard map $E_\alpha(z) = e^{2\pi i\alpha} z e^z$ has a Siegel disk Δ_α and there exists a quasi-conformal homeomorphism $\psi : \mathbb{C} \to \mathbb{C}$ and an E_α–invariant curve Γ in Δ_α such that*

(a) *$\psi(\mathbb{S}^1) = \Gamma$ and ψ maps $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ to the unbounded component, V, of $\widehat{\mathbb{C}} \setminus \Gamma$;*
(b) *ψ conjugates $f : \widehat{\mathbb{C}} \setminus \mathbb{D} \to \widehat{\mathbb{C}}$ to $E_\alpha : \overline{V} \to \widehat{\mathbb{C}}$;*
(c) *$\partial\psi/\partial\bar{z} = 0$ a.e. on $\widehat{\mathbb{C}} \setminus \bigcup_{n\geq 0} f^{-n}(\mathbb{D})$ (in particular ψ is conformal in the interior of this set).*

Observe that this proposition relates, at least qualitatively, the members of the Arnold family to the members of the semistandard one. More precisely, it relates all maps in $\mathcal{T}_\alpha^{\mathrm{AL}}$ to the single map E_α. Any Herman ring of rotation number α can be used by this procedure to produce a Siegel disk with the same rotation number.

Remark 2.2. The surgery construction also connects a different unrelated problem for the two families. It is an open problem to find a parameter value α, if it exists, for which E_α has an unbounded Siegel disk (this is a phenomenon which does occur for the exponential family, for example). With this proposition, this becomes equivalent to finding parameter values (λ, a) such that the Herman ring of $f_{\lambda,a}$ contains the essential singularities in its boundary. See [DF] for further discussion.

We proceed to sketch the surgery construction.

Proof of Proposition 2.1. For this proof, let us take the standard annulus normalized in a different way by setting $\mathcal{A}_r = \{z \in \mathbb{C} \mid 1/r < |z| < r\}$ for $r > 1$.

Let $\phi : \mathcal{A}_r \to H$ be a conformal map that conjugates the rigid rotation $\mathcal{R}_\alpha : \mathcal{A}_r \to \mathcal{A}_r$ to $f : H \to H$. Notice that ϕ must be symmetric with respect the unit circle and hence it leaves $\mathbb{S}^1$ invariant.

We now extend ϕ quasiconformally to the unit disk. Denote by $\hat{\phi} : \mathbb{D}_r \to H \cup \mathbb{D}$ a quasiconformal mapping that agrees with ϕ on $\mathbb{D}_r \backslash \mathbb{D}$, maps $\mathbb{D}$ onto $\mathbb{D}$, and fixes 0.

Define a new map $\hat{f} : \mathbb{C} \to \mathbb{C}$ by

$$\hat{f} = \begin{cases} f & \text{on } \mathbb{C} \backslash \mathbb{D}; \\ \hat{\phi} \circ \mathcal{R}_\alpha \circ \hat{\phi}^{-1} & \text{on } \overline{\mathbb{D}}. \end{cases}$$

The map $\hat{f} : \mathbb{C} \to \mathbb{C}$ is a quasiregular mapping with an essential singularity at infinity. It has one critical point (the one of f that is not inside the disk).

The map $\hat{f}$ is not holomorphic on $\mathbb{D}$, but there it preserves the complex structure defined by the Beltrami form

$$\mu = \frac{\bar{\partial} \hat{\phi}^{-1}}{\partial \hat{\phi}^{-1}}.$$

Pulling back this Beltrami form via $\hat{f}$, we see that there exists a Beltrami form $\hat{\mu}$ that coincides with μ on $\mathbb{D}$, vanishes on $\mathbb{C} \backslash \bigcup_{n \geq 0} \hat{f}^{-n}(\mathbb{D})$ and that is invariant by $\hat{f}$, in the sense

$$\hat{f}^* \hat{\mu} = \hat{\mu}.$$

By the Measurable Riemann Mapping Theorem, there exists a quasiconformal homeomorphism $\psi : \mathbb{C} \to \mathbb{C}$ which fixes 0, sends ω (the critical point) to -1 and such that

$$\hat{\mu} = \frac{\bar{\partial} \psi}{\partial \psi}.$$

Then, the map $\psi \circ \hat{f} \circ \psi^{-1} : \mathbb{C} \to \mathbb{C}$ is an entire transcendental map with one critical point at -1, which fixes 0 and is conjugate to the rotation $\mathcal{R}_\alpha$ in a neighborhood of 0. One can see with some further argument (see [G] or [FSV]) that such a map must be the semistandard map, i.e.,

$$E_\alpha = \psi \circ \hat{f} \circ \psi^{-1}.$$

The map ψ is the required conjugacy. □

To prove the estimate in Theorem 4 one needs to make this surgery construction more explicit and quantitative. The idea is to redo the procedure for the rescaled Arnold family depending on the parameter a and make each of the steps explicit; for example, finding a convenient extension $\hat{\phi}_a$ to the disk so that one can compute its coefficient K_a of quasiconformality. The key part of the proof is finding a good estimate for the quantity $(\psi_a \circ \hat{\phi}_a)'(0)$ in terms of the parameter a (where we are using the notation in the proof above).

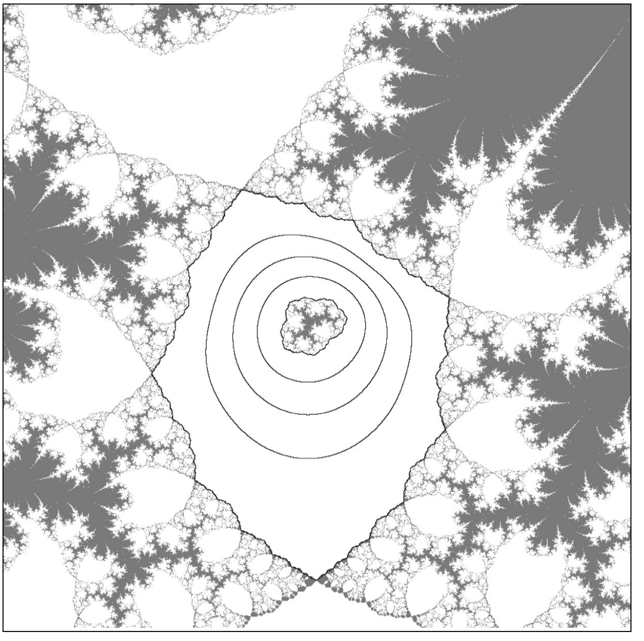


Figure 4: Herman ring of rotation number equal to the golden mean, in the dynamical plane of $f_{\lambda,a}$, where $(\lambda, a) = (e^{2\pi i 0.622359931841}, 0.5i)$. Range: $[-5, 5] \times [-5, 5]$.

3. Complex parameter space. In this section we complexify the parameter space, to improve the bounds obtained in the previous section. The analog of the analytically linearizable part of an (irrational) Arnold tongue is an Arnold disk, which we show to be a disk holomorphically embedded in the parameter space.

Consider the family $\{f_{\lambda,a}\}$ for $\lambda \in \mathbb{C}^*$ and $a \in \mathbb{C}$. Even if λ is not contained in the unit circle and a is not real, $f_{\lambda,a}$ may have a fixed Herman ring. If this is the case there is no reason why it should be symmetric with respect to the unit circle and, indeed, this certainly does not seem to be the case in Figures 4 and 5.

When a vanishes $f_{\lambda,a}$ becomes a linear map $z \mapsto \lambda z$ and we consider that map to have a Herman ring of infinite modulus when $\lambda = e^{2i\pi\alpha}$ and α is a Brjuno number.

Definition 3.1. Suppose $\alpha \in \mathcal{B}$. Let

$$\mathcal{D}_\alpha = \{(\lambda, a) \in \mathbb{C}^* \times \mathbb{C} : f_{\lambda,a} \text{ has a Herman ring of rotation number } \alpha\}.$$

We call $\mathcal{D}_\alpha$ an *Arnold disk.*

The reason for the choice of the name Arnold disk is given by the following theorem that shows that Arnold disks indeed are disks embedded in $\mathbb{C}^2$.

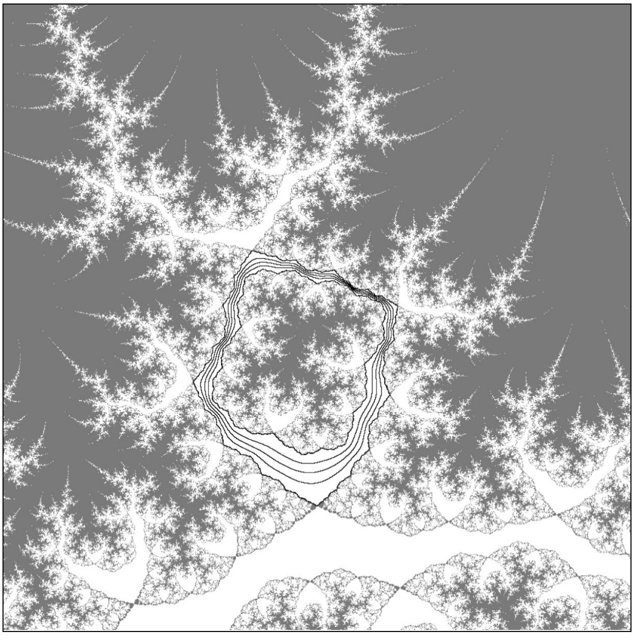


Figure 5: Herman ring of rotation number equal to the golden mean, in the dynamical plane of $f_{\lambda,a}$, where $(\lambda, a) = (e^{2\pi i 0.642219660059}, i)$. Range: $[-4, 4] \times [-4, 4]$.

Theorem 5. *Let α be an arbitrary Brjuno number and denote by r_α the conformal radius of the Siegel disk of the semistandard* map E_α. *The set $\mathcal{D}_\alpha$ is the image of the unit disk $\mathbb{D}$ under an injective holomorphic mapping*

$$\mathcal{F}_\alpha : \mathbb{D} \to \mathcal{D}_\alpha.$$

This mapping can be taken to satisfy the following.

(a) $\mathcal{F}_\alpha(0) = (e^{2i\pi\alpha}, 0)$, *and* $\mathcal{F}'_\alpha(0) = (0, 2r_\alpha)$;

(b) *letting* $\mathcal{F}_\alpha(\delta) = (\lambda(\delta), a(\delta))$, *we have that λ is even and a is odd, i.e., for all $\delta \in \mathbb{D}$*

$$\lambda(-\delta) = \lambda(\delta), a(-\delta) = -a(\delta);$$

(c) *for all $\delta \in \mathbb{D}$, the modulus $m(\delta)$ of the Herman ring of $f_{\mathcal{F}(\delta)}$ satisfies*

$$m(\delta) = \frac{1}{\pi} \log \frac{1}{|\delta|};$$

(d) *for some $\epsilon > 0$, $\mathcal{F}_\alpha(\mathbb{D}_\epsilon)$ is the graph of a holomorphic* map $a \mapsto \lambda(a)$;

(e) *as $|a| \to 0$ the modulus $m(a)$ of the Herman ring of $f_{\lambda(a),a}$ satifies*

$$e^{\pi m(a)} = \frac{2r_\alpha}{|a|} + \mathcal{O}(a).$$

Part (d) is a corollary of a more general result by Risler (see [A]). Notice that part (e) is an improvement of the estimate obtained in the previous section.

We will not prove the properties in the order they are stated. First we see that (d) and (e) follow from the previous three properties. Indeed, part (d) immediately follows from (a), (b) and the implicit function theorem. To see (e) we first note that it follows from (a) that $a(\delta) = 2r_\alpha\delta + \mathcal{O}(\delta^2)$. Since a is an odd function of δ we get $a(\delta) = 2r_\alpha\delta + \mathcal{O}(\delta^3)$, and by the inverse function theorem $\delta(a) = \frac{a}{2r_\alpha} + \mathcal{O}(a^3)$. Combining this fact with (c) we get

$$e^{\pi m(a)} = \frac{1}{|\delta(a)|} = \frac{2r_\alpha}{|a|} + \mathcal{O}(a).$$

Hence to prove the theorem we need to construct the mapping $\mathcal{F}_\alpha$, and establish properties (a), (b) and (c). To do so it is convenient to work with the family $g_{\lambda,b}(w) = \lambda w e^w e^{-b/4w}$, where $\lambda \in \mathbb{C}^*$ and $b \in \mathbb{C}$. The map $w = \frac{a}{2}z$ conjugates $f_{\lambda,a}$ to $g_{\lambda,b}$ with $b = a^2$. Another advantage of working with the g family is that we get rid of the symmetry $f_{\lambda,a}(-z) = -f_{\lambda,-a}(z)$. When $b = 0$ the map $g_{\lambda,b}$ is the semistandard map $w \mapsto \lambda w e^w$ and we adopt the convention that the Siegel disk of $g_{\lambda,0}$ is a Herman ring of (one sided) infinite modulus when $\lambda = e^{2i\pi\alpha}$ and α is a Brjuno number.

We define the analogue of the Arnold disk for the $g_{\lambda,b}$ family as follows.

$$\mathcal{D}'_\alpha = \{(\lambda,b) \in \mathbb{C}^* \times \mathbb{C} : g_{\lambda,b} \text{ has a Herman ring with rotation number } \alpha\}$$

Following our convention $\mathcal{D}'_\alpha$ always contains the point $(e^{2i\pi\alpha}, 0)$.

We state the analog of Theorem 5.

Proposition 3.2. *There exists a holomorphic injection $\mathcal{G}_\alpha : \mathbb{D} \to \mathbb{C}^* \times \mathbb{C}$ that maps the unit disk onto $\mathcal{D}'_\alpha$, and satisfies*

(a′) $\mathcal{G}_\alpha(0) = (e^{2i\pi\alpha}, 0)$;
(c′) *for all $\delta \in \mathbb{D}$, the modulus $m(\delta)$ of the Herman ring of $g_{\mathcal{G}(\delta)}$ satisfies*

$$m(\delta) = \frac{1}{2\pi}\log\frac{1}{|\delta|}.$$

Before giving the proof of the proposition we define an invariant called the twist coordinate of the Herman ring. This is most easily done when the two boundary components of the Herman ring H of $g_{\lambda,b}$ are quasicircles, each containing a critical point. (This is the case when α is of bounded type.) Now, there is a conformal isomorphism $\phi : H \to A_r$ where A_r is the round annulus $\{r < |z| < 1\}$. This map extends as a homeomorphism $\overline{H} \to \overline{A_r}$. We can take this isomorphism to map the outer boundary to the outer boundary and the critical point there to 1. The inner critical point is then mapped to a point $re^{2i\pi\Theta}$. We call the number $\Theta \in \mathbb{R}/\mathbb{Z}$ the twist coordinate of $g_{\lambda,b}$.

In general we cannot assume that ∂H consists of two quasicircles each containing a critical point. But since the boundary components are contained in the closure of the

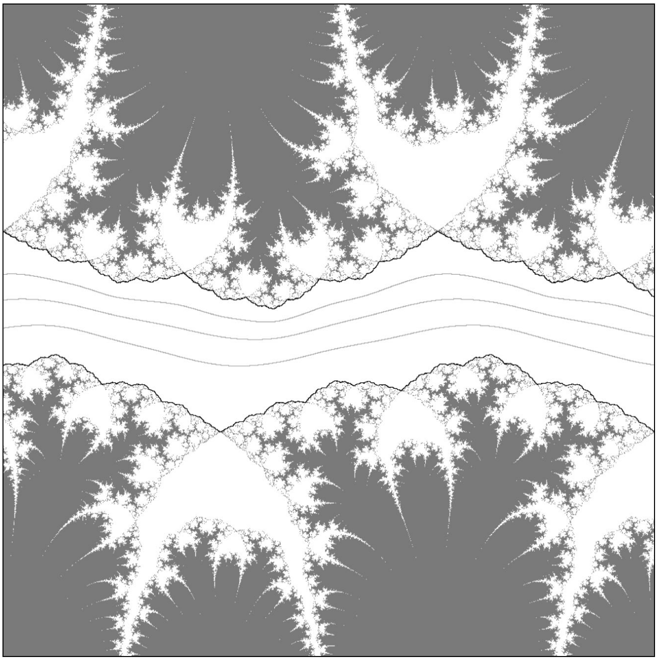


Figure 6: Lift of a Herman ring in the dynamical plane of $z \mapsto z + t + \frac{a}{2\pi}\sin(2\pi z)$. Compare to Figure 4.

forward orbits of the critical points, they are made up of dynamically marked points, and we can still measure to what extend one boundary is twisted with respect to the other one (see [BFGH] for details).

When b is real and positive and $|\lambda| = 1$ then reflection in the circle with center at the origin and radius $\sqrt{b}/2$ conjugates $g_{\lambda,b}$ to itself, and in this case it is easy to check that the twist parameter equals zero.

Figures 6 and 7 show two examples of Herman rings with a twist coordinate of $1/2$ and rotation number equal to the golden mean. The drawings are computed in the dynamical plane of the lift of $f_{\lambda,a}$, that is, $z \mapsto z + t + \frac{a}{2\pi}\sin(2\pi z)$, in order to observe the symmetries better. In fact, these two pictures correspond, once projected back, to the two rings in Figures 4 and 5.

Proof of Theorem 3.2. A portion of the proposition can be deduced from [McS]. Indeed, from their results it can be shown that each component of $\mathcal{D}'_\alpha$ is a pointed disk. Here we show that there is only one component and that the puncture corresponds to $g_{\exp(2i\pi\alpha),0}$. This is done by an explicit construction.

We will give a rough sketch of how to construct the mapping $\mathcal{G}_\alpha$. To give an idea of the mapping, we first describe the inverse map $\Pi : \mathcal{D}'_\alpha \to \mathbb{D}$. The modulus of $\Pi(\lambda, b)$

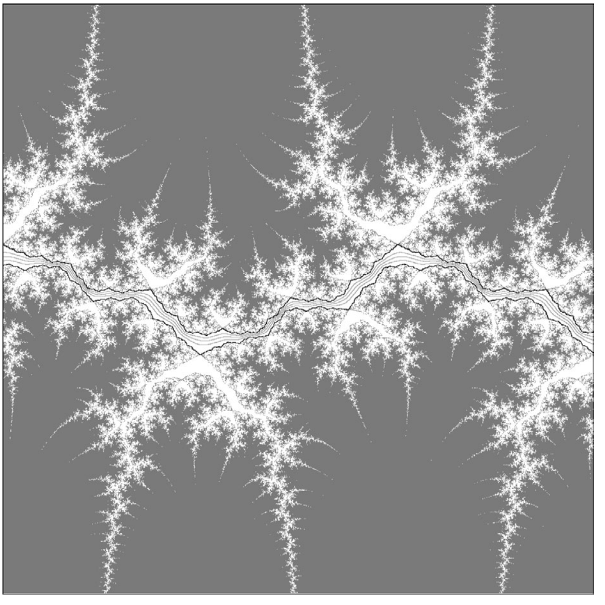


Figure 7: Lift of a Herman ring in the dynamical plane of $z \mapsto z + t + \frac{a}{2\pi}\sin(2\pi z)$. Compare to Figure 5.

is given in terms of the modulus m of the Herman ring of $g_{\lambda,b}$ and the argument is determined by the twist coordinate. More precisely

$$\Pi(\lambda,b) = \exp(-2\pi m + 2i\pi\Theta).$$

We give an outline of the construction of the map $\mathcal{G}_\alpha$. For the details (and there are quite a few), refer to [BFGH]. First we choose a base point g_{δ_0} with a Herman ring H with the desired rotation number. The mapping is produced by changing the complex structure on H and its preimages, as we did in the proof of Theorem 3. This time, we not only change the modulus of the ring but also introduce a twist of one boundary with respect to the other one. In this way, for each $\delta \in \mathbb{D}^*$ we obtain a new member of the family $g_{\lambda(\delta),a(\delta)}$ with a Herman ring whose modulus and twist coordinate are

$$m(\delta) = \frac{1}{2\pi}\log\frac{1}{|\delta|}$$
$$\Theta(\delta) = \frac{1}{2\pi}\arg(\delta).$$

This defines the mapping $\mathcal{G}_\alpha$ from $\mathbb{D}^*$ to the parameter space, satisfying property (c′). Since the Herman ring separates 0 and one critical point from ∞ and the other critical point we can deduce that when δ tends towards 0, then b tends toward 0 as well. By a surgery construction one can show that if $g_{\lambda,b}$ has a Herman ring and b is small then λ is close to $e^{2i\pi\alpha}$. It follows that the constructed mapping extends past the puncture as required in (a′). Finally one shows that the construction does not depend on the choice of base point and that Π indeed is an inverse.

Let us now finish the proof of Theorem 5. We need to prove properties (a), (b) and (c). Notice that the mapping $(\lambda,a) \mapsto (\lambda,a^2) : \mathcal{D}_\alpha \to \mathcal{D}'_\alpha$ provides a two to one covering map ramified at $(\exp(2i\pi\alpha), 0)$ above $(\exp(2i\pi\alpha), 0)$. Hence, there exists an injective holomorphic map $\mathcal{F}_\alpha : \mathbb{D} \to \mathcal{D}_\alpha$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{f_\alpha} & \mathcal{D}_\alpha \\ {\scriptstyle \delta \mapsto \delta^2}\Big\downarrow & & \Big\downarrow {\scriptstyle (\lambda,a) \mapsto (\lambda,\, a^2)} \\ \mathbb{D} & \xrightarrow[\mathcal{G}_\alpha]{} & \mathcal{D}'_\alpha . \end{array}$$

This mapping is unique if we require that the second coordinate of $\mathcal{F}_\alpha(\delta)$ is real and positive when δ is real and positive. Letting $\mathcal{F}_\alpha(\delta) = (\lambda(\delta),\, a(\delta))$ we get from the diagram that $(\lambda(\delta),\, a(\delta)^2) = \mathcal{G}_\alpha(\delta^2)$. It follows that $\lambda(-\delta) = \lambda(\delta)$ so λ is even. It also follows that $a(\delta)^2 = a(-\delta)^2$ so a is either even or odd. Then a has to be odd, because otherwise it would contradict that $\mathcal{F}_\alpha$ is injective. We have proven property (b). Since the Herman ring of $\mathcal{F}_\alpha(\delta)$ is conformally isomorphic to the Herman ring of $\mathcal{G}_\alpha(\delta^2)$ property (c) immediately follows from property (c') in Proposition 3.2.

So to conclude we need only show that $\mathcal{F}'_\alpha(0) = (\lambda'(0),\, a'(0)) = (0,\, 2r_\alpha)$. That $\lambda'(0) = 0$ follows immediately from the fact that λ is even. On one hand, we know from Theorem 1 that

$$\frac{2}{a}(r_\alpha + o(1)) = e^{\pi m(a)},$$

for $a > 0$. On the other hand, we have from (c) that

$$e^{\pi m(a(\delta))} = \frac{1}{\delta}.$$

Combining these two facts we get $a(\delta) = 2r_\alpha\delta + o(\delta)$. This proves (a) and finishes the proof of Theorem 5.

4. The slice of twist coordinate equal to $\frac{1}{2}$.

As it was already mentioned, in general it is not easy to locate complex parameters $(\lambda,a) \in \mathbb{C}^2$ for which the Arnold map $f_{\lambda,a}$ has a Herman ring. The main reason is that, as we saw, these parameters live in surfaces in $\mathbb{C}^2$ isomorphic to disks, one for each fixed rotation number.

There are two exceptional cases where it is not so difficult to locate these parameter values. The first one is the "real" or symmetric case, i.e., when the Herman rings are symmetric with respect to the unit circle or, equivalently, the case where the twist coordinate is equal to 0. Two facts make the computation easier: first, we know that the unit circle is always an invariant curve in the ring, which allows us to compute the rotation number of the map; and second and most important, all these parameters lie on the plane (or cylinder) $\{(\lambda, a) \in \mathbb{R}/\mathbb{Z} \times [0, 1)\}$. Consequently we can apply, for example, bisection methods to locate parameter values for which the ring exists and has a given rotation number (see Section 5 for details).

The second exceptional case turns out to be the slice for which the twist parameter is equal to $1/2$. That is, the two boundary components of the Herman rings are rotated half a turn with respect to each other (see Section 3). Although the symmetry is broken in this case (there is another kind of symmetry which we will describe later) we still have the important property of having these parameter values located on a plane of $\mathbb{C}^2$, namely $\{(\lambda, a) \in \mathbb{R}/\mathbb{Z} \times i\mathbb{R}\}$. This is exactly what we show in the following proposition.

Proposition 4.1. *Suppose $f_{\lambda,a}$ has a Herman ring. Then, the twist coordinate equals $1/2$ if and only if $\lambda \in \mathbb{R}/\mathbb{Z}$ and $a = i\tilde{a}$ with $\tilde{a} \in \mathbb{R}$.*

Proof. For the proof we shall use again the rescaled Arnold family $g_{\lambda,\,b}$. Fix a rotation number $\alpha \in \mathcal{B}$. Recall from Theorem 3.2 that $\mathcal{G}_\alpha$ defines a holomorphic bijection between $\mathbb{D}$ and $\mathcal{D}'_\alpha$, such that $g_{\lambda(\delta),\,b(\delta)}$ has a Herman ring with twist coordinate given by the argument of δ. More precisely,

$$\Theta(\delta) = \frac{1}{2\pi} \arg(\delta).$$

The image by $\mathcal{G}_\alpha$ of the interval $[0, 1)$ is exactly the piece of Arnold tongue $\mathcal{T}_\alpha^{\mathrm{AL}}$, since it follows from Theorem 3 and the injectivity of $\mathcal{G}_\alpha$ that those are the only maps with Herman rings having twist coordinate equal to 0. But now let us look at the map $\mathcal{G}_\alpha$ restricted to the interval $(-1, 1)$. By holomorphy, both components, $\lambda(\delta)$ and $a(\delta)$, must be real analytic. The first one, $\delta \mapsto \lambda(\delta)$ maps $(0, 1)$ into S^1 or equivalently, $\delta \mapsto t(\delta)$ maps $(0, 1)$ into the reals (where $\lambda = e^{2\pi i t}$). It follows that the Taylor series of $t(\delta)$ must have real coefficients and hence, the whole image $t(-1, 1)$ must be real. The same argument shows that $b(-1, 1)$ must also be real.

We conclude that parameters (λ, b) for which the Herman of $g_{\lambda,\,b}$ has twist parameter one half (i.e., the image of $(-1, 0)$ under $\mathcal{G}_\alpha$) lie in $\mathbb{S}^1 \times \mathbb{R}^-$.

To return to the non rescaled Arnold family, recall that $\mathcal{F}_\alpha(\delta) = (\lambda(\delta), a(\delta))$ where $(\lambda(\delta), a(\delta)^2) = \mathcal{G}_\alpha(\delta^2)$. By lifting we deduce that $a(-1, 0) \in i\mathbb{R}$ and $\lambda(-1, 0) \in \mathbb{S}^1$.

To see the other implication, suppose that $f_{\lambda,a}$, $\lambda \in \mathbb{S}^1$, $a \in i\mathbb{R}$ has a fixed Herman ring H. The map $f_{\lambda,a}$ has a symmetry $f_{\lambda,a}(-\frac{1}{\bar{z}}) = -\frac{1}{\overline{f_{\lambda,a}(z)}}$. So H is symmetric with respect to $\tau(z) = -\frac{1}{\bar{z}}$.

We claim that a linearizing map $\psi : H \to \{\frac{1}{r} < |z| < r\}$ will have this symmetry as well. Indeed $\mathcal{R}_\alpha$ and $f_{\lambda,\alpha}$ commute with τ, and hence the map $\psi := \tau\psi\tau : H \to \{\frac{1}{r} < |z| < r\}$ is another linearizing map of H. With this normalization, such maps are unique up to post composition with a rigid rotation, thus $\psi = \mathcal{R}_\theta\psi$ for some $\theta \in [0, 2\pi)$. Now

$$\psi\tau = \tau\mathcal{R}_\theta\psi = \mathcal{R}_\theta\tau\psi.$$

Hence $\psi = \mathcal{R}_\theta \tau \psi \tau = \mathcal{R}_{2\theta}\psi$. It follows that $2\theta = 0 \bmod 1$ or equivalently, that $\theta = 1/2$ or $\theta = 0$. But the first option is not possible because in such a case, $\mathcal{R}_\theta = \tau$ on the unit circle and therefore $\psi\tau = \psi$ on the equator of H, i.e., on $\psi^{-1}(\mathbb{S}^1)$. This would contradict with the injectivity of ψ and hence $\theta = 0$. We have then proved that $\psi\tau = \tau\psi$.

Observe that, as a consequence, every marked point in the boundary of H will have the same property, from which we conclude that the twist coordinate must be $1/2$. □

Although the symmetry with respect to the unit circle (or to the real line in the lift) is lost for maps in this slice (where $|\lambda| = 1$ and $a = i\tilde{a},\ \tilde{a} \in \mathbb{R}$), we just saw that another symmetry appears. Indeed, it is easy to check that

$$f_{\lambda,i\tilde{a}}\left(-\frac{1}{\bar{z}}\right) = -\frac{1}{\bar{f}_{\lambda,i\tilde{a}}(c)}$$

and that the lift $F_{t,\,i\tilde{a}}(z) = z + t + i\,\frac{\tilde{a}}{2\pi}\sin(2\pi z)$ satisfies

$$F_{t,i\tilde{a}}(\bar{z} + \pi) = \overline{F_{t,i\tilde{a}}(z)} + \pi.$$

As a consequence, the two critical points of the lift (seen in the cylinder) which are located at

$$\omega_1 = \frac{\pi}{2} - i\ \operatorname{arcsinh}\left(\frac{1}{\tilde{a}}\right) \ \text{and}\ \ \omega_2 = \frac{3\pi}{2} + i\ \operatorname{arcsinh}\left(\frac{1}{\tilde{a}}\right)$$

do not have independent dynamics (as in the general case). It then makes sense to compute a $(t, \tilde{a})$–plane picture where we check if the orbit of the critical point escapes to infinity or otherwise remains bounded. The result is shown in Figure 8, where we have also superposed the rational tongues of rotation number $1/4$, $1/2$ and $3/4$ and the irrational curve corresponding to rotation number equal to the golden mean (see Section 5.4 for the algorithms).

Figures 4 and 5 show the dynamical plane for two of the parameters in the irrational curve while Figures 6 and 7 correspond to the lifts of these.

We observe from Figure 8 that many of the rational tongues do not seem to cross this slice. For example, it is easy to check that none of the maps in this parameter plane can have an attracting fixed point, and therefore, there is no zero – tongue emanating from the point (0, 0). Similarly, there does not seem to be any rational tongue of odd denominator attached to the bottom line $\tilde{a} = 0$.

5. Numerical algorithms. In this section we describe the numerical algorithms used to create the pictures in the paper. The methods are quite general and may be used to compute the same type of pictures for other families possessing a cycle of Herman rings (or Siegel disks) as long as it is the only existing periodic Fatou cycle.

We start by assuming we already know the parameter values for which the map has a Herman ring H. Later on we shall see how to compute them, but first we see how to draw a dynamical plane picture with these given parameter values.

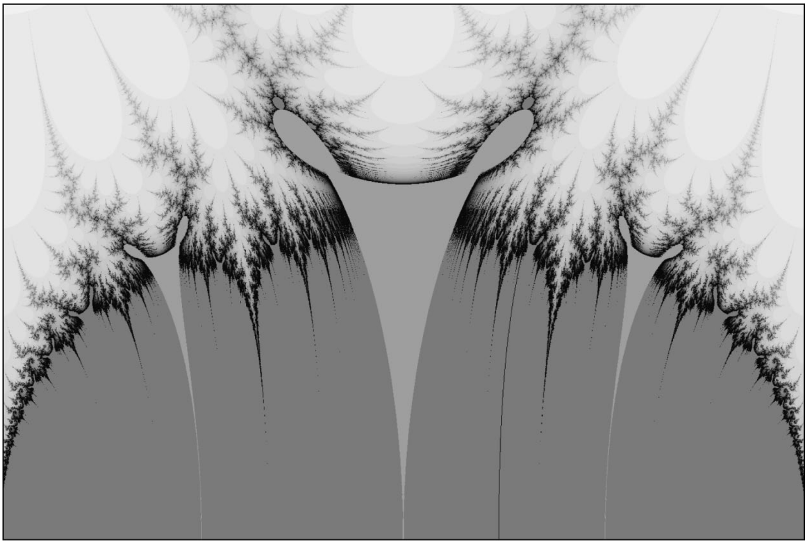


Figure 8: $(t, \tilde{a})$ − parameter plane where the map $f_{\lambda,ia}$ is iterated to check if the critical orbits seem to remain bounded (dark grey). Range: $[0, 1] \times [0, 2]$. All Herman rings of maps in this slice have twist parameter one half. Superposed, we find the rational tongues of rotation numbers $1/4$, $1/2$ and $3/4$ and the irrational curve corresponding to rotation number equal to the golden mean. See Section 5.4.

Escaping algorithms usually work poorly for holomorphic maps of $\mathbb{C}^*$ (i.e., those with essential singularities at 0 and ∞). It is common for their Julia set to have positive measure and it therefore appears very thick. Moreover, exponentiating repeatedly makes overflows and underflows appear too often and too soon.

The algorithms used here to draw the dynamical planes are of a different nature. Given a pixel, we ask wether the corresponding center point eventually falls inside the Herman ring, in which case it is painted in white. Pixels which do not satisfy this property are painted in color.

To be able to answer this key question we must first find what we call a *base domain* of the Herman ring, i.e., a set A inside the ring satisfying the following: every point in the Herman ring has an orbit which eventually intersects A. These base domains are of a different shape depending on the map we work with.

In all cases, we use the following important fact: the orbit of the critical points accumulates on the boundary of the Herman ring. Hence, we always can compute two lists of points that correspond to the critical orbits. These points are drawn in the picture so the boundary of the ring is outlined.

5.1. Symmetric Herman rings (Figure 2). In this case we look for a base domain in the form of a true annulus around the unit circle, since we know that the unit circle

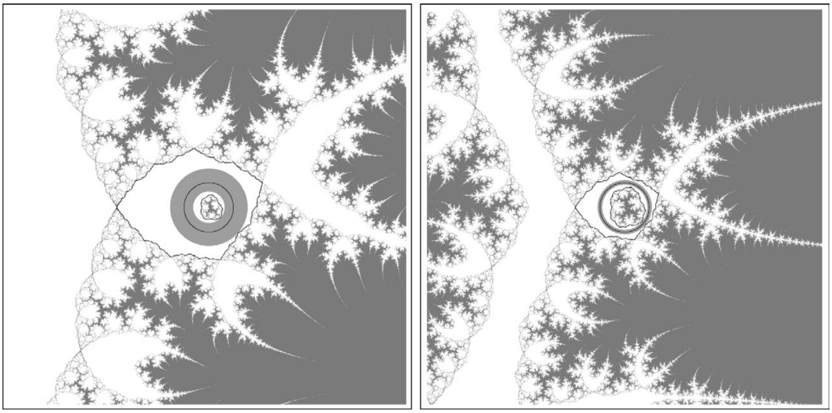


Figure 9: Herman rings with rotation number equal to the golden mean, for the map $f_{\lambda,a}$ for chosen parameters $(t, a) = (0.614526385907, 0.5)$ (left) and $(t, a) = (0.610074404161, 0.8)$ (right) where $\lambda = \exp(2\pi i t)$. The base annulus is drawn inside each of the rings. Range: $[-8, 8] \times [-8, 8]$. Compare to Figure 2.

is always completely contained in the ring. To find the width of the annulus we use the symmetry of the ring. Indeed, we find the point on the outer boundary which has the smallest modulus, say $r > 1$. Necessarily, the reflexion of this point with respect to the unit circle is the point on the inner boundary with the largest modulus, $\frac{1}{r}$. Then every orbit of H meets the annulus $A = \{\frac{1}{r} < |z| < r\}$ and therefore A is a base domain. See Figure 9.

5.2. Siegel disk (Figure 3). If the invariant Fatou component is a Siegel disk with center p, we may look for a base domain in the form of a disk, centered also at p. To find its radius, we choose from all the points on the boundary of the disk (i.e., the critical iterates) the one that is closest to the center of the disk, say at distance r. The disk $D(p, r)$ is a base domain, and all invariant circles must cross the radial segment that joints p with the closest point on the boundary. See Figure 10.

5.3. Non-symmetric Herman rings (Figures 4 and 5). This is the hardest case since we do not know a priori the location of the Herman ring in the dynamical plane, nor any orbit (or any point, for that matter) contained inside the ring. We do have, however, the lists of points that correspond to the approximated boundaries of H. We will choose a base domain in the form of a round disk with the condition of being entirely contained in H and touching both boundaries. To do this, we need to find the minimum (at least a local minimum) of the distance from points on the outer boundary to points on the inner boundary. One can do this, for example, by picking a point in the first list (outer), say p_1; then finding the closest point to p_1 in the other list (inner), say

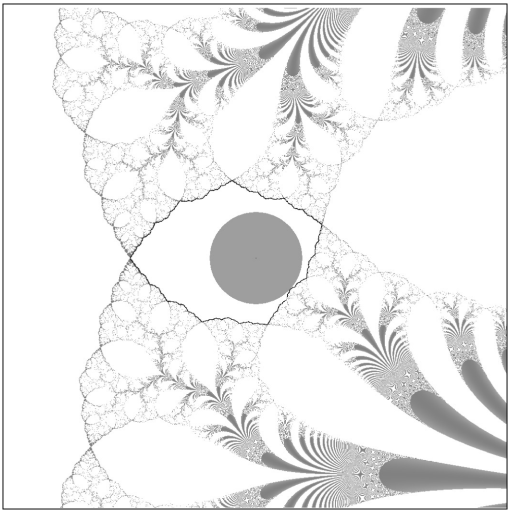


Figure 10: Siegel disk of the function $E_\alpha(z) = e^{2\pi i\alpha} z e^z$, with rotation number α, equal to the golden mean. The base disk has been drawn inside the disk. Range: $[-2, 2] \times [-2, 2]$.

q_1; then the closest point to q_1 in the first list, say p_2, etc. We stop once the points do not change any more, and hence we have a pair (p_n, q_n) whose distance between each other is at least a local minimum in the following sense: no point in the inner boundary is closer to p_n than q_n; and viceversa, no point in the outer boundary is closer to q_n

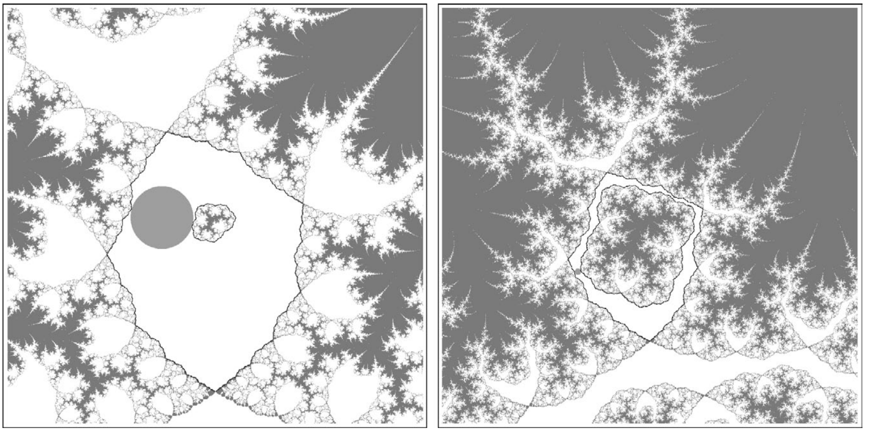


Figure 11: Herman rings of rotation number equal to the golden mean, in the dynamical plane of $f_{\lambda,a}$, where $(\lambda, a) = (e^{2\pi i 0.622359931841}, 0.5i)$ (left) and $(\lambda, a) = (e^{2\pi i 0.642219660059}, i)$ (right). The base disks has been drawn inside the ring.

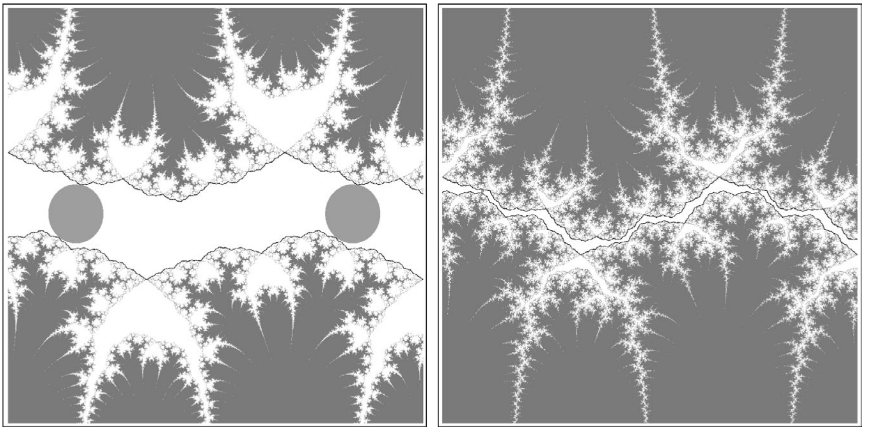


Figure 12: Lifts of the Herman rings in Figure 11, in the dynamical plane of $z \mapsto z + t + \frac{a}{2\pi}\sin(2\pi z)$. The base rings have been drawn inside the Baker domain.

than p_n. This assures that the disk of radius $|p_n - q_n|/2$ centered at the middle point between p_n and q_n is completely contained in H and touches the two boundaries. See Figure 11. A similar computation can be made for the lift of these rings as it is shown in Figure 12.

5.4. Parameter space: finding parameters for a given rotation number. To draw the images in this paper, we needed to locate parameters in a given parameter slice for which $f_{\lambda,a}$ has a Herman ring with a given rotation number, say α. We observe that if f_{λ_0,a_0} has a Herman ring with rotation number α of bounded type, then the quantity

$$\rho^n_{t_0,a_0}(\omega_{t_0,a_0}) = \frac{F^n_{t_0,a_0}(\omega_{t_0,a_0}) - \omega_{t_0,a_0}}{n}$$

has limit α, where $\omega_{t_0,a_0} \in \partial H$ is a critical point and $\lambda_0 = e^{2\pi i t_0}$. Also, if $f_{t,a}$ has an attracting periodic cycle, then $\rho^n_{t,a}(\omega_{t,a})$ tends to a rational number. Hence we expect $\rho^N_{t,a}$ for a given N large, to be close to a real number for a substantial part of parameter space.

Suppose we can find (t_0,a_0), (t_1,a_1) such that $\rho^N_{t_0,a_0}$ and $\rho^N_{t_1,a_1}$ are close to being real and

$$\mathrm{Re}(\rho^N_{t_0,a_0}) < \alpha < \mathrm{Re}(\rho^N_{t_1,a_1}).$$

We then pick points randomly on the segment between (t_0,a_0) and (t_1,a_1) until we find (t_2,a_2) such that $\rho^N_{t_2,a_2}$ is close to being real, and replace one of the previous pairs by this pair, as in the classical bisection procedure. If we manage to find parameters for which $\rho^N_{t,a}$ is almost real, the continuity of $\rho^N_{t,a}$ guarantees the convergence of $\rho^N_{t_n,a_n}$ towards α, if the length of the segments decreases to 0.

We do not claim this to be a fullproof method; it is a heuristic one that seems to work reasonably, especially for initial values (t_0,a_0), (t_1,a_1) with $\mathrm{Re}(a_0)$ and $\mathrm{Re}(a_1)$ small.

References

[A] V. I. ARNOL'D, *Small denominators. I: Mappings of the circumference onto itself*, Am. Math. Soc., Transl., II. Ser. **46**, (1965) 213-284; translation from Izv. Akad. Nauk SSSR, Ser. Mat. 25, 21–86 (1961).

[BFGH] X. BUFF, N. FAGELLA, L. GEYER, C. HENRIKSEN, Arnold Disks and Herman Rings, *Preprint* www.fagella.maia.ub.es/fagella/papers.html.

[Bru] A.D. BRUNO, *Analytic forms of differential equations*, Trans. Mosc. Math. Soc. **25** (1971).

[dMvS] W. DE MELO, S. VAN STRIEN, *One–Dimensional Dynamics*, Springer–Verlag, (1991).

[DF] P. DOMÍNGUEZ, N. FAGELLA, Existence of Herman rings for meromorphic functions, *Complex Variables* (To appear).

[F] N. FAGELLA, Dynamics of the complex standard family, *J. of Math. Anal. and Appl.* **229** (1999), 1–31.

[FG] N. FAGELLA, L. GEYER, *Surgery on Herman rings of the complex standard family*, Ergod. Th. & Dynam. Sys., **23**, 493–508 (2003).

[FSV] N. FAGELLA, T. M. SEARA, J. VILLANUEVA, *Asymptotic Size of Herman Rings of the Complex Standard Family by Quantitative Quasiconformal Surgery*, To appear in Ergod. Th. & Dynam. Sys.

[G] L. GEYER, SIEGEL DISCS, Herman Rings and the Arnold family, *Trans. Amer. Math. Soc.* **353:9** (2001), 3661–3683 (electronic).

[Her] M. HERMAN, Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations, *Inst. Hautes Études Sci. Publ. Math.* **49** (1979), 5–233.

[McS] C. MCMULLEN, D. SULLIVAN *Quasiconformal homeomorphisms and dynamics. III: The Teichmüller space of a holomorphic dynamical system*, Adv. Math., **135**, No. 2, 351–395 (1998).

[Ri] E. RISLER, *Linéarisation des perturbations holomorphes des rotations et applications*, Mém. Soc. Math. Fr., **77**, (1999).

[Rü] H. RÜSSMANN, Kleine Nenner. II. Bemerkungen zur Newtonschen Methode, *Nachr. Akad. Wiss. Göttingen Math.–Phys. Kl. II* (1972), 1–20.

[S] M. SHISHIKURA, *On the quasiconformal surgery of rational functions*, Ann. Sci. Éc. Norm. Sup., **vol** 20, 1–29, (1987).

[Y] J. C. YOCCOZ, Analytic linearization of circle diffeomorphisms, *Dynamical systems and small divisors (Cetraro, 1998)*, 125–173, Lecture Notes in Math. **1784**, Springer, Berlin, 2002.

Departament de Matemàtica Aplicada i Anàlisi,
Universitat de Barcelona,
Gran Via 585, 08007 Barcelona (Spain)
fagella@maia.ub.es

Department of Mathematics
Technical University of Denmark
Matematiktorvet, Building 303,
DK-2800 Kgs. Lyngby (Denmark)
chris@mat.dtu.dk

Stretching rays and their accumulations, following Pia Willumsen

Tan Lei

Dedicated to Bodil Branner's 60th birthday

Abstract. Pia Willumsen did her graduate work with Bodil Branner. Willumsen has written a very beautiful Ph.D. thesis, [W], containing many interesting results. They have, however, remained unpublished up to today. We present here a short account of some of Willumsen's results, sketch some of the proofs, as well as some immediate extensions. The main topic is *stretching rays*, which is the analogue in higher dimensional parameter space of external rays of the Mandelbrot set. In the space of cubic polynomials, the interaction of the two critical points create new and interesting phenomena. A typical case deals with maps with a parabolic basin containing both critical points. The results we present here provide necessary conditions for accumulation and landing of stretching rays to these maps. Also, discontinuity of the wring operator on the cubic (in contrast to quadratic) polynomials, is proven.

1. Definitions and statements. All polynomials in this article will be monic and centered, of degree greater than or equal to 2.

Let P be such a polynomial. Let μ be a P-invariant Beltrami form with $||\mu||_{L^\infty} \le 1$. It induces a family of P-invariant Beltrami forms $t \cdot \mu$ for t running through the unit disk $\mathbb{D}$. Obviously $||t \cdot \mu||_{L^\infty} \le |t| < 1$. One can thus apply the Measurable Riemann Mapping theorem to this t-analytic family of Beltrami forms.

For $t \in \mathbb{D}$, we will define χ_t to be the integrating map of $t \cdot \mu$ normalized so that $\chi : \mathbb{D} \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a holomorphic motion and that the new dynamics $P_t := \chi_t \circ P \circ \chi_t^{-1}$ is again a monic centered polynomial for each t.

Thus the pair (P, μ) induces an (analytic, see e.g. [PT]) family of quasi-conformal deformations of P. We are interested in the boundary behavior of such deformations.

A fundamental choice for μ ([W, §5]) is the following: denote by φ_P the Böttcher coordinates defined on a neighborhood U of ∞, normalized to be tangent to the identity at ∞, and by $B(\infty)$ the basin of ∞, define

$$\mu_P(z) := \begin{cases} (\log \circ \varphi_P \circ P^n)^* \frac{d\bar{z}}{dz} & \text{for } z \in B(\infty) \text{ and for large } n \text{ such that } P^n(z) \in U \\ 0 & \text{for } z \notin B(\infty) \end{cases}$$

One can check easily that the definition is independent of the choice of n. In this case the holomorphic motion on the dynamical plane $\chi : \mathbb{D} \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is called **Branner-Hubbard motion**, and the induced operator $S : (t, P) \mapsto P_t$ on the parameter space is called **wring operator**.

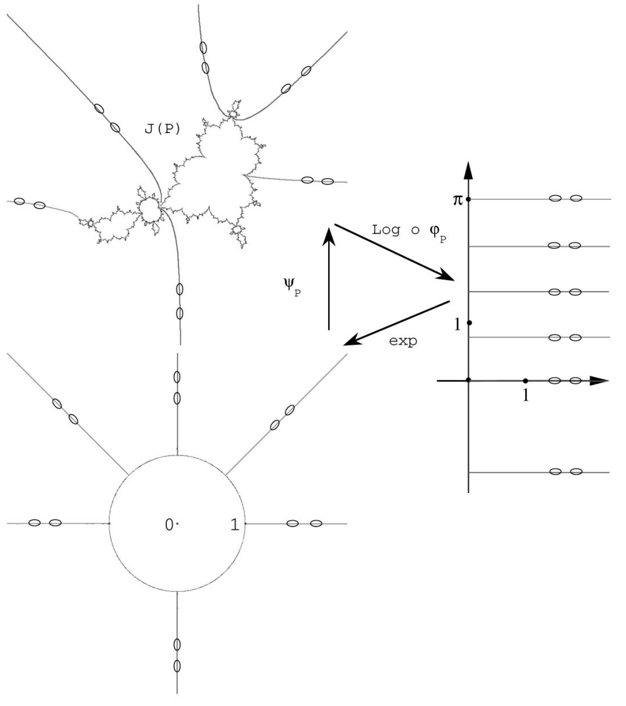


Figure 1: Ellipses of $t\mu_P$ in the dynamical plane, in the Böttcher coordinates and in the log-Böttcher coordinates.

Geometrically, $t\mu_P$ corresponds to an ellipse field supported on the basin of ∞ of P, with constant ellipticity, and with the major axis of the ellipse tangent to the external rays when $-1 < t < 0$ but orthogonal to the external rays when $0 < t < 1$. See Figure 1. As $-1 < t < 0$ and $t \searrow -1$, the ratio of major to minor axis tends to ∞. In this case, as the corresponding integrating map χ_t maps these ellipses to circles, it therefore 'pushes' the points closer to the filled Julia set along the external rays.

The real locus $S(P) = \{S(t,\, P),\, t \in]-1,\, 1[\}$ is called the **Stretching ray** (or in short S-ray) through P. As $t \searrow -1$, the eventual escaping critical points of P_t get closer and closer to the filled Julia set, so that the polynomials P_t accumulate to the connectedness locus of polynomials of the same degree. If an escaping critical point of P sits on a periodic external ray, one should expect a creation of parabolic points as $t \searrow -1$ in order to capture this critical point.

It is quite easy to check that for P in the quadratic family $z \mapsto z^2 + c$ with disconnected Julia set, the stretching ray through P is exactly the external ray through P of the exterior of the Mandelbrot set. And if the escaping critical point of P sits on the

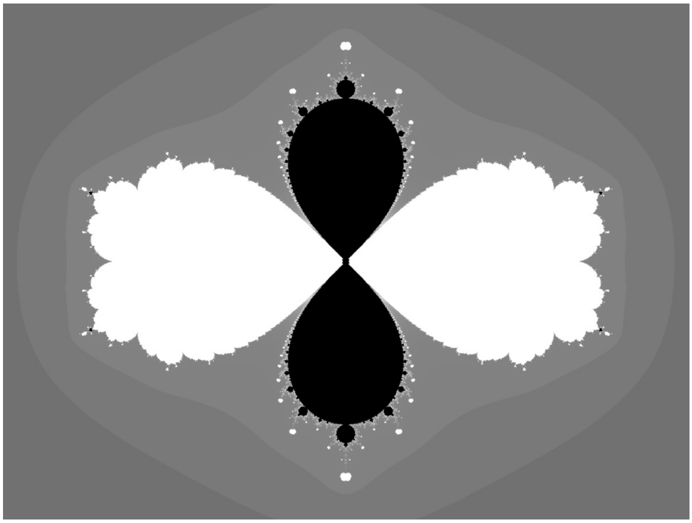


Figure 2: The slice $Per_1(1)$ and its subset $\mathcal{A}$, which is the two large white butterfly wings.

0-external ray, the stretching ray P_t converges to the cauliflower polynomial $Q(z) = z^2 + \frac{1}{4}$.

In general, a stretching ray may, or may not, land on a point of the connectedness locus. One may thus study the accumulation points of such a ray. This paper focuses on necessary conditions on the pairs (P, Q) such that $S(P)$ accumulates to Q and such that creation of parabolic points occur at Q.

In the quadratic case, if $S(P)$ accumulates to the cauliflower polynomial $Q(z) = z^2 + \frac{1}{4}$, then the dynamical 0-external ray of P must branch at the critical point, consequently $S(P)$ coincides with the 0-external of the Mandelbrot set and $S(P)$ actually lands at Q. This non-trivial fact is closely related to the local connectivity of the Mandelbrot set at $\frac{1}{4}$, and has several proofs in the literature. See for example [T], and the remark below.

The situation in the cubic case is considerably more complicated, due the presence of the two critical points. And this is precisely the focus of the present study.

Denote by $\mathcal{A}$ the set of cubic polynomials such that both critical points are contained in the same immediate basin of a parabolic fixed point of multiplier 1 (therefore with a Jordan curve Julia set and all other periodic points repelling). These polynomials are called **cubic cauliflowers**. Two examples of such Julia sets can be found on the right column of Figure 5. And the parameter set $\mathcal{A}$ is represented, in some suitable parametrization, in Figure 2, as well as in the middle picture of Figure 5.

The main purpose here is to study necessary conditions on a pair (P, Q) of cubic polynomials so that Q belongs to $\mathcal{A}$ and is accumulated by $S(P)$. We study also the continuity or non-continuity of the map $(t, P) \mapsto S(t, P)$.

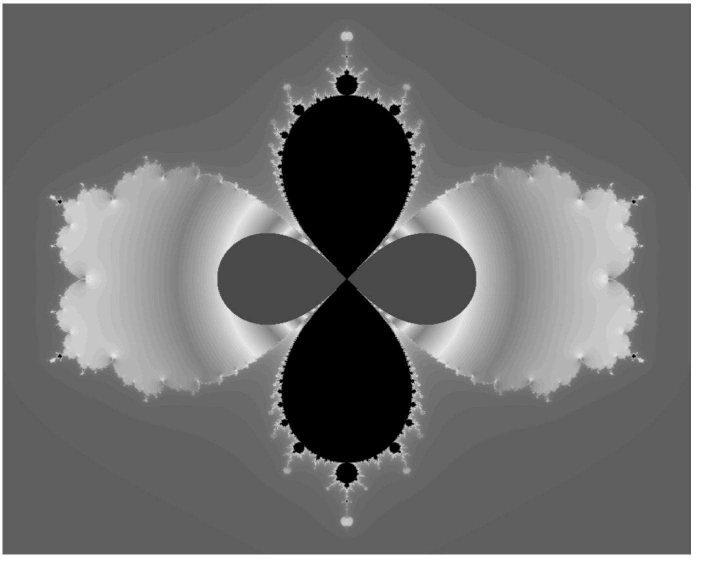


Figure 3: The lemniscate in $\mathcal{A}$ indicates the parabolic attracting locus.

Following A. Epstein, a polynomial $Q \in \mathcal{A}$ is called *parabolic attracting* if any nearby map has an attracting or parabolic fixed point. Such polynomials are represented in Figure 3.

Here is the first main result: denote by $\mathrm{Acc}(S(P))$ to be the set of accumulation points of $S(P)$ as $t \searrow -1$.

Theorem 1.1. *Let (P, Q) be a pair of cubic polynomials such that $Q \in \mathrm{Acc}(S(P)) \cap \mathcal{A}$ and $Q \neq P$. Then*

I. *All periodic points of P are repelling and Q is not parabolic attracting* ([W, 6.5]).
II. *The filled Julia set K_P of P is a Cantor set* ([W, 7.13]).
III. (*exchange* 0 *and* $\frac{1}{2}$ *if necessary*) *The* 0*-external ray of P branches at a critical point of P* ([W, 7.5]); *the left and right limit* 0*-ray* (*see definition below*) *land at two distinct fixed points of P, which collide to the parabolic fixed point of Q at the limit; the* $\frac{1}{2}$*-ray of P lands at the third fixed point, and at any other periodic point of P lands exactly one external ray, which is periodic.*
IV. *For the other critical point of P, either*
   a) *it escapes and falls eventually into the* 0*-external ray; or*
   b) *it is the landing or branching point of a θ-external ray, with an angle θ satisfying $\{3^k\theta, k \in \mathbb{N}\} \not\ni 0$ but $\overline{\{3^k\theta, k \in \mathbb{N}\}} \ni 0$.*

Conjecturally Case b) *never occurs.*

Two examples of the pair (P, Q) can be found on the top and bottom row of Figure 5. The branching 0-external ray for P, and the 0-external ray for Q are also drawn.

Remark. As mentioned above, there is a similar statement in the quadratic setting (the part IV is void). Only part III is non trivial. Most of the existing proofs in the literature depend on the global combinatorial structure of the quadratic family. But the proof of Part III presented here can be easily adapted to give a purely intrinsic proof of its quadratic parallel.

A theorem of Branner-Hubbard shows the continuity of the wring operator restricted either to the cubic connectedness locus or to the cubic disconnectedness locus ([BH1, §9]), in fact it acts trivially on the connectedness locus. However, the second main result presented here claims:

Theorem 1.2. *The* map $(t, P) \mapsto S(t, P)$ *is continuous in the space of quadratic polynomials, but is discontinuous on the cubics, more precisely at some points in* $\mathbb{D} \times \mathcal{A}$ ([W, 5.7, 5.8]).

Further necessary conditions on landing or accumulation of S-rays to $\mathcal{A}$ in terms of Lavaurs maps will be given in §4, as well as the proof of Theorem 1.2. The proof of Theorem 1.1 is in §3.

The following concept and result will be interesting for further research in the topic, but will not be needed nor proved in the present paper. For a given (polynomial, invariant Beltrami form) pair (P, μ), and χ_t the suitably normalized integrating maps of $t\mu$, the associated initial speed of χ_t and the **ground wind** at P relative to μ are defined by

$$\tau(z) := \frac{d}{dt}\chi_t(z)\bigg|_{t=0}, \quad w(P, \mu) := \frac{d}{dt}P_t\bigg|_{t=0}.$$

Proposition 1.3 ([W, 5.18])**.** *Assume* μ *agrees with* μ_P *on the basin of infinity. We then have*

$$w(P, \mu) = \tau \circ P - P' \cdot \tau.$$

2. Necessary conditions for accumulation.

2.1. Generalized external angles and rays. We need to generalize the notion of external rays, angles etc. to polynomials with disconnected Julia set. Let P be such a polynomial, say monic and centered of degree 3. There is a unique univalent map φ_P, the Böttcher coordinate, defined at least in a neighborhood of ∞, conjugating P to $z \mapsto z^3$ and satisfying $\frac{\varphi_P(z)}{z} \to 1$ as $z \to \infty$. Denote by ψ_P the inverse map of φ_P. Then the maximal domain of definition of ψ_P is the complement of $\overline{\mathbb{D}} \cup Y_P$, where Y_P is the union of finitely many radial segments $\{[1, r_{\theta_i}] \cdot e^{2\pi i\theta_i}\}$ together with their successive preimages under the iteration of z^3, and ψ_P extends continuously to the tips of segments in Y_P. For $\theta \in \mathbb{T}$ define the θ-ray by

$$R_P(\theta) = \psi_P(\{]r_\theta, +\infty[\cdot e^{2\pi i\theta}\})$$

where r_θ is the minimal value possible. It is said to be *branched* with branching point $\psi_P(r_\theta \cdot e^{2\pi i\theta})$ if $r_\theta > 1$; *unbranched* otherwise. It *lands* at a Julia point a if $a = \lim_{r\searrow 1} \psi_P(re^{2\pi i\theta})$. Due to the countability of radial segments in Y_P, the following limits exist and are called **left and right limit θ-ray**:

$$R_P^{\pm}(\theta) = \bigcup_{r>1} \{ \lim_{u\to 0^{\pm}} \psi_P(re^{2\pi i(\theta+u)}) \}.$$

See the middle top and middle bottom pictures in Figure 5 for examples of left and right limit 0-rays of cubic polynomials.

These limit rays are canonically parametrized by $r > 1$ and preserved by P, in the sense that $P(R_P^+(\theta)) = R_P^+(3\theta)$ and $P(R_P^-(\theta)) = R_P^-(3\theta)$. Now we define the *external angles* $\arg(a)$ and the *generalized external angle* $\operatorname{Arg}(a)$ by:

$$a \in K_P \begin{cases} \arg(a) := \{\theta \mid R_P(\theta) \text{ lands at } a\} \\ \cap \\ \operatorname{Arg}(a) := \{\theta \mid R_P^+(\theta) \text{ or } R_P^-(\theta) \text{ lands at } a\} \end{cases}$$

$$a \notin K_P \begin{cases} \arg(a) := \{\theta \mid a \in R_P(\theta) \cup \{\text{branching point}\}\} \\ \cap \\ \operatorname{Arg}(a) := \{\theta \mid a \in R_P^+(\theta) \cup R_P^-(\theta)\}. \end{cases}$$

Clearly

$$3 \cdot \operatorname{Arg}(a) \subset \operatorname{Arg}(P(a)). \tag{1}$$

Some of these sets might be empty. In case that K_P is connected, we have $Y_P = \emptyset$, $R_P^{\pm}(\theta) = R_P(\theta)$ for every θ and $\arg(a) = \operatorname{Arg}(a)$ for every a.

Define also

$$\text{per} - \arg(a) := \{\theta \text{ periodic} \mid R_P(\theta) \text{ lands at } a\}. \tag{2}$$

2.2. Fundamental properties of accumulation of a stretching ray. Let (P, Q) be a pair of monic centered polynomials of same degree, with $Q \in \operatorname{Acc}(S(P))$. Let $t_n \searrow -1$ be a sequence such that the polynomials $P_n := S(t_n, P)$ converges algebraically to Q. We compare here critical points and periodic points of P, P_n and Q.

(Most results below are also valid under the more general assumption that $P_n \to Q$ algebraically and P_n are qc-conjugates of a single map P, not necessarily coming from stretching).

Let α be a periodic point of P. It is associated with a triple (m, λ, k), with:

- $m =$ period of $=: \text{per}(\alpha)$;
- $\lambda := (P^m)'(\alpha) =$ the multiplier of α; and
- $k =$ the multiplicity of α, as root of $P^m(z) - z = 0$.

Recall that χ_{t_n} is a qc-conjugacy from P to P_n. Let $\alpha_n := \chi_{t_n}(\alpha)$ denote the corresponding periodic point for P_n. Its associated triple (m_n, λ_n, k_n) satisfies $m_n = m$ and $k_n = k$, but λ_n might be depending on n.

Taking a subsequence if necessary, we may assume $\alpha_n \to \alpha'$. Clearly α' is again a periodic point of Q, of period a divisor of m. We denote its associated triple by (m', λ', k'). Thus $m'|m$. We look for further relations between these triples. We look also for possible relations between critical orbits of P and those of Q.

We group the various notations in one tableau:

See Table 1 for a summery of some of the following result.

(m, λ)	$(m_n = m, \lambda_n)$		(m', λ')	(period, multiplier)
α	$\alpha_n = \chi_{t_n}(\alpha)$	$\overset{sub\text{-}sequence}{\longrightarrow}$	α'	periodic point
P	$S(P) \ni P_n$	$\longrightarrow$	Q	
w	$w_n = \chi_{t_n}(w)$	$\overset{sub\text{-}sequence}{\longrightarrow}$	w_Q	w escaping critical point
θ	$\theta_n = \theta$		?	escaping critical angle

Proposition 2.1. *In the above setting,*

1. *The escaping critical angles for P are preserved along $S(P)$, as well as the ratio of escaping critical potential levels (i.e., real part in the* $\log \circ \varphi_Q$*-coordinate).*
2. *More generally a point z and $\chi_t(z)$ have the same set of generalized external angles.*
3. *If $|\lambda| \le 1$ and $\lambda \neq 1$, then $(m, \lambda, k) = (m', \lambda', k') = (m, \lambda, 1)$* ([W, 7.1]).
4. *If $\lambda' = 1$ then $m' = m$, and either $|\lambda| > 1$ or $\lambda = 1$. On the other hand, if $\lambda = 1$ then either $m = m'$ and $\lambda' = 1$, or $m > m'$, $\lambda' \neq 1$ and $\lambda'^{\frac{m}{m'}} = 1$* ([W, 7.2]).
5. *Any critical relation of P is preserved to Q (maybe with a divisor period).*
6. *The* map *Q has a connected Julia set. All rational rays of Q land. For every parabolic or repelling periodic point z' of Q, we have* $\arg_Q(z')$ *non empty and consisting of only periodic angles (see* [Mi] *and* [Pe]). *If P has connected Julia set then $S(P) = \{P\}$ and $\chi_t|_{K_P} = id$* ([BH1, 8.3], *see also Lemma* 4.1.a) *below).*
7. *If $|\lambda'| < 1$ then $(m, \lambda, k) = (m', \lambda', k') = (m, \lambda, 1)$. Consequently if α is repelling then α' can't be attracting.*
8. *If $|\lambda'| > 1$ then $|\lambda| > 1$ and $(m, k) = (m', k') = (m, 1)$ (but maybe $\lambda \neq \lambda'$).* Moreover, for the non-generalized external angles:
$$\mathrm{per} - \arg_P(\alpha) = \mathrm{per} - \arg_{P_n}(\alpha_n) \supset \arg_Q(\alpha') \neq \emptyset.$$
Furthermore, the inclusion $\supset$ becomes an equality iff for any $\theta \in \mathrm{per} - \arg_P(\alpha)$, the ray $R_Q(\theta)$ for the polynomial Q does not land at a parabolic point of Q.
9. *If a periodic ray $R_P(\theta)$ of P branches then the corresponding ray $R_Q(\theta)$ for Q lands at a parabolic periodic point (see Figure 5 for examples).*

Proof. By uniform convergence, $Q^m(\alpha') = \alpha'$. So $m'|m$ and

$$\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} (P_n^m)'(\alpha_n) = (Q^m)'(\alpha') = ((Q^{m'})'(\alpha'))^{\frac{m}{m'}} = (\lambda')^{\frac{m}{m'}}. \tag{3}$$

Points 1 and 2 are due to the definition of the wring operator.

Point 3. The fact that $m_n = m$ is due to the global qc-conjugacy χ_{t_n}. We prove at first $\lambda_n \equiv \lambda$ for $|\lambda| \le 1$. If $|\lambda| < 1$, α is an attracting periodic point and the complex structure is not deformed in the attracting basin. So χ_{t_n} is a local conformal conjugacy and $\lambda_n = \lambda$. If $|\lambda| = 1$, the map χ_{t_n} is a topological conjugacy and the multiplier is preserved by a (highly non-trivial) theorem of Naishul (see [Na] or [P-M]).

Assume now $|\lambda| \le 1$ and $\lambda \ne 1$. Then $(\lambda')^{\frac{m}{m'}} = \lambda$ by the above argument and lambda, so $\lambda' \ne 1$. By Rouché's theorem, in a neighborhood of α' and for n large, P_n^m has a unique fixed point u_n, and $P_n^{m'}$ has a unique fixed point v_n. But $m'|m$. So $P_n^m(v_n) = v_n$. By unicity, $u_n = v_n = \alpha_n$. But $\mathrm{per}(v_n) \le m' \le m = \mathrm{per}(\alpha_n)$. It follows that $m' = m$ and $\lambda' = \lambda$. The part $k = k' = 1$ is easy.

Point 4. Assume $\lambda' = 1$. By Point 3, either $|\lambda| > 1$ or $\lambda = 1$. We want to show $m' = m$. Set $\alpha' = 0$ for simplicity. Let k be the multiplicity. The local expansions of $Q^{m'}$ and Q^m are:

$$Q^{m'}(z) = z + az^k + \cdots; \quad Q^m(z) = z + \frac{m}{m'}\, az^k + \cdots; \quad a \ne 0.$$

Applying Rouchés theorem again, we get that in a neighborhood of α' and for n large, P_n^m has exactly k fixed points (counting with multiplicity) $u_n^1, u_n^2, \ldots, u_n^k$ (one of them must be α_n). And $P_n^{m'}$ has exactly k fixed points (counting with multiplicity) $v_n^1, v_n^2, \ldots, v_n^k$. But $m'|m$. So $v_n^i = \alpha_n$ for some i. But $\mathrm{per}(v_n^i) \le m' \le m = \mathrm{per}(\alpha_n)$. It follows that $m' = m$.

On the other hand, assume $\lambda = 1$. Then $\lambda_n \equiv 1$ and $(\lambda')^{\frac{m}{m'}} = 1$ by the proof in Point 3. If $m' = m$ then $\lambda' = 1$. If $m' < m$ then $\lambda' \ne 1$ by the previous paragraph.

Point 5 is easy.

Point 6. The map Q has no escaping critical points, so has a connected Julia set. The rest are proved in the given references.

Point 7. $|\lambda'| < 1 \Longrightarrow |\lambda_n| < 1 \Longrightarrow \lambda = \lambda_n = \lambda'$ and $m' = m$ (due to Point 3).

Point 8. $|\lambda'| > 1 \Longrightarrow |\lambda_n| > 1 \Longrightarrow |\lambda| > 1$ due to Points 3 and 4. The fact that $m' = m$ is because $\lambda \ne 1$. Now Point 6 gives that $\arg_Q(\alpha') \ne \emptyset$ and consists of only periodic angles. The set $\{\alpha'\} \cup \bigcup_{\theta \in \arg(\alpha')} R_Q(\theta)$ undergoes a holomorphic motion. So for $\theta \in \arg_Q(\alpha')$, for P_n close to Q, and for α_n the perturbed periodic point, $R_{P_n}(\theta)$ continues to land at α_n. But $\arg_P(\alpha) = \arg_{P_n}(\alpha_n)$. So $\mathrm{per} - \arg_P(\alpha) \supset \arg_Q(\alpha')$.

Finally for $\theta \in \mathrm{per} - \arg_P(\alpha)$, we know that θ is periodic, thus the ray $R_Q(\theta)$ does not branch and lands at a repelling or parabolic point z' (Snail Lemma, see for example [P-M]). If z' is repelling then by above there is a periodic point z of P with $\theta \in \arg_P(z)$. As a consequence $z = \alpha$ and $z' = \alpha'$. So $\theta \in \arg_Q(\alpha')$.

Point 9. Again $R_Q(\theta)$ must land and lands at a repelling or parabolic point. The rest follows from Point 8. See Figure 5 for examples of such pairs (P, Q). □

Table 1 summarizes the results in Proposition 2.1 about relations between a polynomial P with disconnected Julia set (on the left) and an accumulation point Q of its stretching ray (on the right).

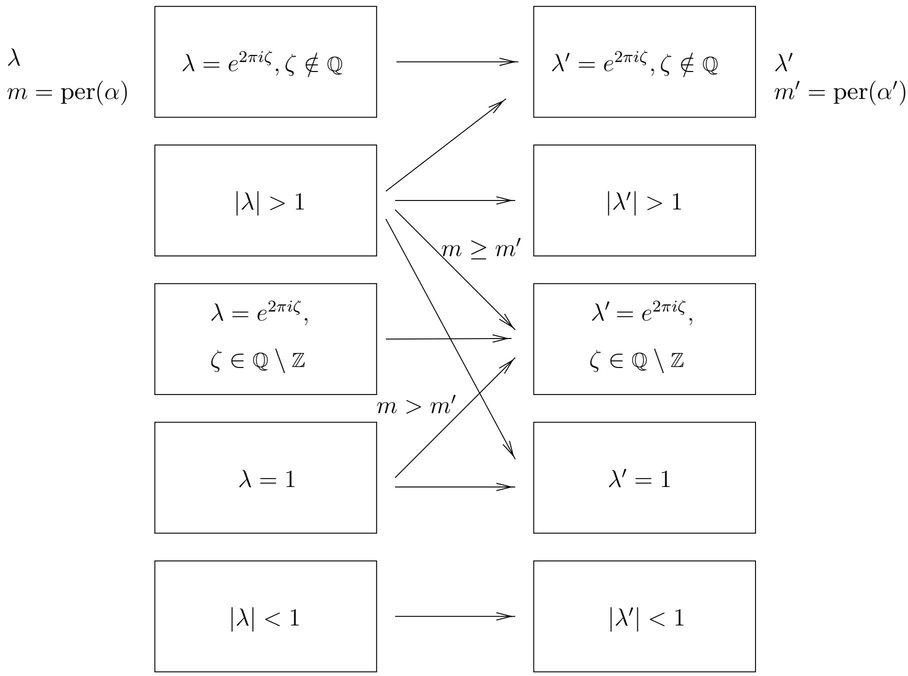


Table 1: In most cases we have $m = m'$. The only exceptions are the two diagonal arrows which are specially labeled.

The next result establishes Part I of Theorem 1.1:

Corollary 2.2. Let (P, Q) be a pair of cubic polynomials such that $P \neq Q \in \mathrm{Acc}(S(P)) \cap \mathcal{A}$. Then all periodic points of P are repelling, and Q is not parabolic attracting.

Proof. Note that P can not be in the connectedness locus, for otherwise $S(P) = \{P\} = \mathrm{Acc}(S(P)) \not\ni Q$ by Proposition 2.1.(6). We use the fact that Q has a unique non-repelling periodic point β', whose associated triple is $(1, 1, 2)$, i.e., it is fixed with multiplier 1, and the multiplicity of β' as a root of $Q(z) - z = 0$ is 2 (this follows from the assumption that both critical points of Q are contained in the same attracted basin of β', so that β' has only one Fatou petal).

Let α be a periodic point of P, of associated triple (m, λ, k). We want to prove that $|\lambda| > 1$.

As indicated at the beginning of this section, and due to Proposition 2.1, the point α induces a periodic point α' of Q, whose associated triple (m', λ', k') is related to those of α following Table 1. On the other hand, we know from the property of Q that either $|\lambda'| > 1$ or $\alpha' = \beta'$ and $\lambda' = 1$.

We conclude immediately that either $|\lambda| > 1$ or $\lambda = 1$ by Table 1.

Assume $\lambda = 1$. By Table 1, either $m = m'$ and $\lambda' = 1$, or $m > m'$ and λ' is a non-trivial root of unity. Again the latter case is not possible for our Q. So $m = m'$ and $\lambda' = 1$. This implies that $\alpha' = \beta'$ and $m = m' = 1$. Thus P is in the space $\mathrm{Per}_1(1)$ of cubic polynomials having a fixed point of multiplier 1 and P has a disconnected Julia set. But then $S(P)$ accumulates to the boundary of the connected locus in this space, which is disjoint from $\mathcal{A}$. This is a contradiction to the fact that $Q \in \mathrm{Acc}(S(P)) \cap \mathcal{A}$.

We conclude then $|\lambda| > 1$. Therefore all periodic points of P are repelling.

Now let $P_n \in S(P)$ with $P_n \to Q$. If Q were parabolic attracting, some P_n would have an attracting or parabolic fixed point. On the other hand, P_n is a qc-deformation of P. So, as P, all periodic points of P_n are repelling. This is a contradiction. □

Lemma 2.3. *If a critical point w of P has a rational generalized external angle θ that is not a preimage of the* 0*-angle, then* $\mathrm{Acc}(S(P)) \cap \mathcal{A} = \emptyset$.

Proof. If $w \in \partial K_P$ then it must be strictly preperiodic. And this critical relation is preserved to any $Q \in \mathrm{Acc}(S(P))$, by Proposition 2.1.(5). This implies $Q \notin \mathcal{A}$, for otherwise both critical points of Q would be contained in the attracted basin of a parabolic fixed point and would not be preperiodic.

Assume now $w \notin K_P$. Assume there is $Q \in \mathrm{Acc}(S(P)) \cap \mathcal{A}$. The fact that $Q \in \mathcal{A}$ implies that the θ-external ray of Q lands at a prerepelling point. This ray is therefore stable under perturbation, by Proposition 2.1.(8). This means that for any polynomial sufficiently close to Q, its θ-external ray is unbranched and lands at a prerepelling point, in particular it does not contain critical points. On the other hand, the fact that $Q \in \mathrm{Acc}(S(P))$ implies the existence of a sequence of polynomials P_n converging to Q, such that the θ-external ray of P_n, as that of P, contains a critical point. This is a contradiction. □

3. Proof of Theorem 1.1.

Proof. Assume $P \neq Q \in \mathrm{Acc}(S(P)) \cap \mathcal{A}$. Part I is already proved in Corollary 2.2.

II. If both critical points of P escape, then K_P is a Cantor set (this is classical). If instead only one critical point escapes, then, letting L be the filled-Julia-component containing the non-escaping critical point, either K_P is a Cantor set or L is m-periodic for some m, in which case $P^m|_L$ is hybrid equivalent to $z^2 + c$ for some c in the Mandelbrot set. This is due to Branner-Hubbard theory for cubics ([BH2, 5.3]).

Assume by contradiction that L is m-periodic.

Then the hybrid equivalent quadratic polynomial $z^2 + c$ has a fixed point $\hat{\alpha}$ which is either non-repelling or repelling without 0 as an external angle. This fixed point corresponds to a m-periodic point α for P. Let $P_n = S(t_n, P)$ so that $P_n \to Q$. Then $\alpha_n := \chi_{t_n}(\alpha)$ has a limit α' (taking a subsequence if necessary). Denote by (m', λ', k') the associated triple of α'. It is related to that of α following Table 1.

We use the fact that Q has a unique periodic point β' with associated triple $(1, 1, 2)$, and all other periodic points are repelling. So either $\alpha' = \beta'$ and $(m', \lambda', k') = (1, 1, 2)$, or $|\lambda'| > 1$.

Assume at first $m > 1$. By Table 1 we can not have both $m' = 1$ and $\lambda' = 1$. So $\alpha' \neq \beta'$ and $|\lambda'| > 1$. By Table 1 again we have $m' = m$. As the Julia set of Q is a Jordan curve, $\arg_Q(\alpha')$ consists of a unique angle θ, which has the same period m. By Proposition 2.1.(8), $\theta \in \arg_P(\alpha)$. Now the hybrid conjugacy sends the germ of $R_P(\theta)$ to a fixed external access to $\hat{\alpha}$ for the polynomial $z^2 + c$. This in turn implies (using the Böttcher coordinates for $z^2 + c$) that the 0-external ray lands at $\hat{\alpha}$, contradicting the choice of $\hat{\alpha}$.

Assume now $m = 1$. Assume first $c = \frac{1}{4}$. In this case P is in the $\mathrm{Per}_1(1)$ slice and $S(P)$ accumulates to the boundary of the connected locus in this slice, which is disjoint from $\mathcal{A}$, and thus gives a contradiction. Assume now $\hat{\alpha}$ is an attracting or neutral fixed point (with multiplier $\neq 1$). Then α' has the same property by Table 1, and is again not possible for our Q. Assume finally $\hat{\alpha}$ is repelling with at least two external rays. Then α is repelling for P. By Table 1 the point α' is either repelling (and fixed) or parabolic with multiplier 1. The former case is excluded due to the same argument as in the case $m > 1$. So we are left with the case that $\alpha' = \beta'$ is the unique parabolic fixed point of Q. By Theorem 1.(3–4) in [LP], $\mathrm{Arg}_P(\alpha) = \{\theta_1, \ldots, \theta_p\}$ with $p \geq 1$ and all θ_i periodic. The case $p = 1$ can be treated as above. If $p > 1$, there is one θ_i that is not the 0-angle, and $R_Q(\theta_i)$ lands at a repelling periodic point $z' \neq \alpha'$. Due to the stability of $\overline{R_Q(\theta_i)}$ under small perturbation, for n large, $R_{P_n}(\theta_i)$ is unbranched and lands at a point z_n tending to z'. On the other hand, an unbranched ray is unique, so $z_n = \chi_{t_n}(\alpha) = \alpha_n \to \alpha'$. This is again a contradiction.

III. We group the various notations in the next tableau.

γ		γ_n		γ' a repelling fixed point
$\overline{\bigcup_{\theta \in A_\eta} R_P(\theta)}$		$\overline{\bigcup_{\theta \in A_\eta} R_{P_n}(\theta)}$		$\overline{\bigcup_{\theta \in A_\eta} R_Q(\theta)}$, $\eta > 0$
does not contain		does not contain		does not contain
α, β	χ_{t_n}	α_n, β_n	$n \to \infty$	β' parabolic fixed point
P	$\mapsto$	P_n	$\longrightarrow$	Q
$z \neq \alpha, \beta$		$\chi_{t_n}(z) \neq \alpha_n, \beta_n$		$z' \neq \beta'$

Choose $P_n = S(t_n, P)$ with $P_n \to Q$ as $t_n \searrow -1$. As Q has a Jordan curve Julia set, every external ray lands, and different rays land at distinct points. Recall that β' denotes the parabolic fixed point of Q with multiplier 1. Assume $R_Q(0)$ lands at β' (otherwise exchange 0 with $\frac{1}{2}$). Denote by γ' the landing point of $R_Q(\frac{1}{2})$.

Clearly $\gamma' \neq \beta'$ and γ' is a repelling fixed point. Apply Proposition 2.1.(8) to γ', one proves that $R_P(\frac{1}{2})$ lands at some point γ, and $\chi_{t_n}(\gamma) =: \gamma_n$ converges to γ'.

By part II, K_P is a cantor set. Thus all fixed points of P are simple. There are therefore $\alpha \neq \beta$ two fixed points of P distinct from γ. Set $\alpha_n = \chi_{t_n}(\alpha)$ and $\beta_n = \chi_{t_n}(\beta)$. They are distinct from γ_n and are bounded away from ∞. Further the limit of any convergent subsequence of α_n must be a fixed point of Q, distinct from γ' (as in a neighborhood of γ' there can be only one fixed point of P_n, which is γ_n). But the only fixed points of Q are β' and γ'. Thus the entire sequence α_n converges to β'. We have $\mathrm{Arg}_P(\alpha) = \mathrm{Arg}_{P_n}(\alpha_n)$ due to the stretching properties of χ_t. Similarly for β, β_n. By Corollary 2.2 both α, β are repelling.

Claim 1. The set $\mathrm{Arg}_P(\alpha)$ is non-empty, closed, and satisfies: $3 \cdot \mathrm{Arg}_P(\alpha) \subset \mathrm{Arg}_P(\alpha)$.

Proof. The inclusion is due to the fact that $P(\alpha) = \alpha$ and (1). The rest follows from [LP, 2.1] (or [W, 2.10]) (in fact when K_P is a Cantor set, every left or right limit ray lands, and every Julia point has a non-empty compact set of generalized external angles, by pulling back disks with equipotential boundaries).

Claim 2. We have $0 \in \mathrm{Arg}_P(\alpha)$.

Proof. For any fixed $\eta > 0$ set

$$A_\eta = \{\theta \in \mathbb{T} \mid d_{\mathbb{T}}(3^k\theta, 0) \geq \eta \text{ for all } k\},$$

et let X_η be the set of landing points of $R_Q(\theta)$ for all $\theta \in A_\eta$. Then $\beta' \notin X_\eta \cup \bigcup_{\theta \in A_\eta} R_Q(\theta)$ and X_η is a compact hyperbolic subset ([ST], Theorem 1.2). Furthermore $X_\eta \cup \bigcup_{\theta \in A_\eta} R_Q(\theta)$ undergoes a holomorphic motion over a small neighborhood of Q, preserving the dynamics. Therefore for n large, the ray $R_{P_n}(\theta)$ is unbranched for all $\theta \in A_\eta$ and

$$\alpha_n, \beta_n \notin \overline{\bigcup_{\theta \in A_\eta} R_{P_n}(\theta)}.$$

(And the right hand set does not contain critical points of P_n). So

$$\left(\bigcup_{\eta > 0} A_\eta\right) \cap \mathrm{Arg}_{P_n}(\alpha_n) = \emptyset.$$

The same is true if we replace $\mathrm{Arg}_{P_n}(\alpha_n)$ by $\mathrm{Arg}_P(\alpha)$ as the two sets are equal. But $\mathrm{Arg}_P(\alpha)$ is closed, non-empty and forward invariant by angle tripling (by Claim 1). So

$$\theta \in \mathrm{Arg}_P(\alpha) \Longrightarrow 3^k\theta \in \mathrm{Arg}_P(\alpha) \Longrightarrow \overline{\{3^k\theta, k \in \mathbb{N}\}} \subset \mathrm{Arg}_P(\alpha) \Longrightarrow 0 \in \mathrm{Arg}_P(\alpha),$$

where the last implication is due to the fact that $\theta \notin \bigcup_{\eta > 0} A_\eta$. This ends the proof of Claim 2.

Similarly one shows $0 \in \mathrm{Arg}(\beta)$. So 0 is a generalized external angle for both α and β. This means that the 0-ray of P is branched at a critical point, that $R_P^{\pm}(0)$ land at two distinct fixed points which collide to the parabolic fixed point of Q in the limit.

Now let $z \neq \alpha, \beta$ be any periodic point of P. Let z' be a limit point of $\chi_{t_n}(z)$. It is a periodic point for Q. We now show that $z' \neq \beta'$. If $\mathrm{per}(z) = 1$, then $z = \gamma$ and

$z' = \gamma' \neq \beta'$. So we may assume $m := per(z) > 1$. Either $\mathrm{per}(z') < m$ then z' is parabolic but with multiplier distinct from 1 (Table 1). This is not possible for Q. Or $\mathrm{per}(z') = m > 1$. In this case $z' \neq \beta'$.

As all periodic points of Q other than β' are repelling, so is z'. As $Q|_{J_Q}$ is topologically conjugate to $z^3|_{S^1}$ we have $\arg_Q(z') = \{\theta'\}$ and θ' is m-periodic.

Now let $\theta \in \arg_P(z)$ be periodic, that is the ray $R_P(\theta)$ does not branch and lands at z. Clearly $\theta \neq 0$ as the 0-ray branches. But $R_Q(\theta)$ always lands and lands at a distinct point than the landing point of $R_Q(0)$, which is β'. Therefore $R_Q(\theta)$ lands at a non parabolic point. So the equality in Propostion 2.1.(8) holds for z and z', i.e.,

$$\{\theta \in \arg_P(z),\ \theta \text{ periodic}\} = \{\theta'\}.$$

To prove that $\arg_P(z) = \{\theta'\}$ we just need to apply the (non-trivial) result [LP, Theorem 1.4].

IV. Let w be a critical point of P, escaping or not. The fact that K_P is a Cantor set ensures that $\mathrm{Arg}_P(w) \neq \emptyset$ (see Claim 1). Let $\theta \in \mathrm{Arg}_P(w)$.

Case 1. There is N such that $3^N\theta = 0$. Then w must escape for otherwise $P^{N+1}(w) = P^N(w)$ and this critical relation would have persisted to Q. This is Case a) of the theorem.

Case 2. There is $\eta > 0$ such that $\theta \in A_\eta$. This is not possible due to the argument in III, similar to the proof of Lemma 2.3.

Case 3. The orbit of θ does not meet 0 but accumulates to 0. In this case either $w \notin K_P$, in which case the θ-ray branches at w, or $w \in K_P$ in which case $\mathrm{Arg}(w) = \arg(w) \neq \emptyset$. This is Case b) of the theorem. Conjecturally this case never occurs. □

4. Lavaurs maps, Fatou vectors and Theorem 1.2.

4.1. Lavaurs maps and enriched Branner-Hubbard motion. In this subsection, denote by Q a monic centered polynomial of degree d, with connected Julia set and with a parabolic fixed point β' of multiplier 1. We will define the following objects related to Q:

$$(Q, \beta', B(\beta'), K_Q, B_Q(\infty), \varphi_Q, \psi_Q, \Phi_Q^{\pm}, \Psi_Q^{+}, g_{\tilde{\sigma}}) \tag{4}$$

where:

- $B(\beta')$ is an immediate basin of β';
- K_Q is the filled Julia set
- $B_Q(\infty)$ is the basin of ∞;
- φ_Q denotes the Böttcher coordinates of Q near ∞, tangent to the identity at ∞. As K_Q is connected, the map φ_Q extends to a conformal homeomorphism from $B_Q(\infty)$ to $\mathbb{C}\setminus\overline{\mathbb{D}}$.
- $\psi_Q : \mathbb{C}\setminus\overline{\mathbb{D}} \to B_Q(\infty)$ denotes the inverse of φ_Q.

• Φ_Q^- denotes the attracting Fatou coordinates. More precisely, it is at first defined and univalent on an attracting petal of β' satisfying $\Phi_Q^- \circ Q = T_1 \circ \Phi_Q^-$. It is then extended to the entire basin $B(\beta')$ using the functional equation (and is no more univalent).
• Φ_Q^+ denotes the repelling Fatou coordinates. It is at first defined and univalent on a repelling petal of β' satisfying $\Phi_Q^+ \circ Q = T_1 \circ \Phi_Q^+$.
• Ψ_Q^+ denotes the inverse of this local Φ_Q^+. It is then extended to the entire plane $\mathbb{C}$ using the functional equation $Q \circ \Psi_Q^+ = \Psi_Q^+ \circ T_1$ (and is not globally univalent).
• T_* denotes the translation $z \mapsto z + *$, in particular T_1 is the translation by 1.
• For $\tilde{\sigma} \in \mathbb{C}$, the **Lavaurs map** $g_{\tilde{\sigma}}$ of lifted phase $\tilde{\sigma}$, is by definition $\Psi_Q^+ \circ T_{\tilde{\sigma}} \circ \Phi_Q^-$. It satisfies $g_{\tilde{\sigma}} \circ Q = Q \circ g_{\tilde{\sigma}}$.

The Fatou coordinates $\Phi_Q^\pm$ are uniquely defined up to post-composition of a translation. We may for example choose the following normalizations:

Φ_Q^- has an inverse branch mapping the right half plane univalently onto a region U_Q whose boundary contains β' and a critical point w_0 (in some sense w_0 is the 'closest' attracted critical point), and $\Phi_Q^-(w_0) = 0$;

$\Psi_Q^+(0) = \psi_Q(e^x)$ for some $x > 0$ (in other words $\Psi_Q^+(0)$ is a point on the 0-ray of Q), and Ψ_Q^+ is univalent on the left half plane.

The various maps and change of coordinates are sketched in the following commutative diagram:

$$
\begin{array}{ccccccc}
 & & \mathbb{C}/\mathbb{Z} & \xrightarrow{T_\sigma} & \mathbb{C}/\mathbb{Z} & & \\
 & & \pi\uparrow & & \pi\uparrow & & \\
\mathbb{C}^- & \xrightarrow{T_1} & \mathbb{C}^- & \xrightarrow{T_{\tilde{\sigma}}} & \mathbb{C}^+ & \xrightarrow{T_1} & \mathbb{C}^+ \\
\Phi_Q^-\uparrow & & \Phi_Q^-\uparrow & & \Psi_Q^+\downarrow & & \downarrow\Psi_Q^+ \\
B(\beta') & \xrightarrow[Q]{} & B(\beta') & \xrightarrow[g_{\tilde{\sigma}}]{} & \mathbb{C} & \xrightarrow[Q]{} & \mathbb{C} \\
 & & & & \cup & & \cup \\
 & & & & B_Q(\infty) & \xrightarrow[Q]{} & B_Q(\infty) \\
 & & & & \psi_Q\uparrow & & \uparrow\psi_Q \\
 & & & & \mathbb{C}\setminus\overline{\mathbb{D}} & \xrightarrow[z^d]{} & \mathbb{C}\setminus\overline{\mathbb{D}} \\
 & & & & l_t\downarrow & & \downarrow l_t : z \mapsto z \cdot |z|^{\frac{2t}{1-t}} \\
 & & & & \mathbb{C}\setminus\overline{\mathbb{D}} & \xrightarrow[z^d]{} & \mathbb{C}\setminus\overline{\mathbb{D}}
\end{array}
$$

In the above diagram, the sets $\mathbb{C}^\pm$ are two different copies of the complex plane. The map π denotes the natural projection from $\mathbb{C}$ to $\mathbb{C}/\mathbb{Z}$. For σ the class of $\tilde{\sigma}$ in $\mathbb{C}/\mathbb{Z}$, the map T_σ denotes simply the quotient of $T_{\tilde{\sigma}}$. For $t \in \mathbb{D}$, the map $l_t : z \mapsto z \cdot |z|^{\frac{2t}{1-t}}$, $\mathbb{C}\setminus\overline{\mathbb{D}} \to \mathbb{C}\setminus\overline{\mathbb{D}}$, is a quasi-conformal map commuting with z^d.

We may now describe the Branner-Hubbard motion of Q as follows: One checks easily (see for example [PT]) that the complex structure $t\mu_Q$ defined in §1 coincides with the pull-back of the standard structure by $l_t \circ \varphi_Q$. We use as in §1 the map χ_t to denote the integrating map of $t\mu_Q$, uniquely normalized so that $\chi_t Q \chi_t^{-1}$ is again a monic centered polynomial.

Part a) of the following Lemma in contained in [BH1, 8.3] (we include a sketch of its proof for completeness). The rest part of the following two lemmas are inspired from [PT, §4.2].

Lemma 4.1. *Fix any* $t \in \mathbb{D}$.

a) *We have* $\chi_t \circ Q \circ \chi_t^{-1} = Q$, $\chi_t|_{K_Q} = id$, *and* $\chi_t = id$ *on the ideal boundary of* $B_Q(\infty)$.

b) $\chi_t|_{B_Q(\infty)} = \psi_Q \circ l_t \circ \psi_Q^{-1}$, $\chi_t \circ \psi_Q = \psi_Q \circ l_t$, and $\Phi_Q^- \circ \chi_t|_{B(\beta')} = \Phi_Q^-$.

c) *Denote by* $\mu_t^+ := (\Psi_Q^+)^*(t\mu_Q)$, *then there is a unique integrating map* ξ_t *of* μ_t^+ *such that*

$$\xi_t \circ T_1 = T_1 \circ \xi_t, \quad \chi_t \circ \Psi_Q^+ = \Psi_Q^+ \circ \xi_t. \tag{5}$$

d) *Denote by* $\widetilde{K}(Q)$, *resp.* $\widetilde{B}_Q(\infty)$, *the preimage by* Ψ_Q^+ *of* K_Q, *resp. of* $B_Q(\infty)$. *Then* $\xi_t|_{\widetilde{K}_Q} = id$, *the map* $\Psi_Q^+ : \widetilde{B}_Q(\infty) \to B_Q(\infty)\setminus\{\infty\}$ *is a universal covering, and* $\xi_t = id$ *on the ideal boundary of* $\widetilde{B}_Q(\infty)$.

Proof. a) This part uses only the fact that Q has a connected Julia set, as stated in Proposition 2.1.(6) above. To prove it, define

$$h_t = \begin{cases} \psi_Q \circ l_t \circ \psi_Q^{-1} & \text{on } B_Q(\infty) \\ id & \text{on } K_Q. \end{cases}$$

Then one checks easily, using the explicit formula of l_t, that h_t satisfies a) in place of χ_t, and h_t is a global homeomorphism. One needs to apply Rickman's gluing lemma to prove that h_t is actually quasi-conformal, therefore an integrating map of $t\mu_Q$.

Now $h_t \circ \chi_t^{-1} : B_Q(\infty) \to B_Q(\infty)$ is analytic on z, conjugates Q to itself, equals to the identity for $t = 0$ and depends continuously on t. As there are only finitely many such conformal selfconjugacies we conclude that $h_t = \chi_t$ for all t. This settles a).

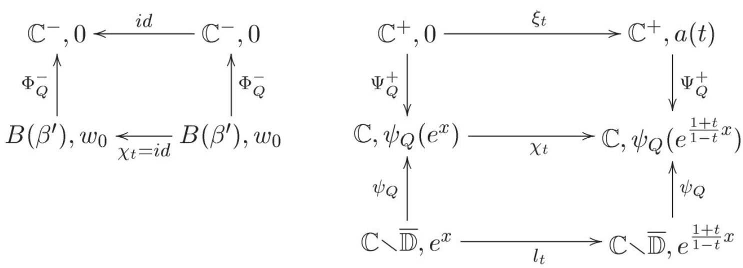
b) The first two equalities follow from the explicit construction of χ_t above. The last one is also trivial as $\chi_t|_{B(\beta')} = id$ by a).

c) As $t\mu_Q$ is Q-invariant, the complex structure μ_t^+ is T_1-invariant. Denote by ζ_t the unique integrating map of μ_t^+ fixing 0, 1, ∞. Then $\zeta_t \circ T_1 \circ \zeta_t^{-1}$ is a global conformal homeomorphism of $\mathbb{C}$, is fixed point free, and maps 0 to 1 by the normalization of ζ_t. So $\zeta_t \circ T_1 \circ \zeta_t^{-1} = T_1$.

Now $\chi_t \circ \Psi_Q^+ \circ \zeta_t^{-1}$ is analytic in z and conjugates T_1 to Q. So it is a repelling Fatou coordinate. By unicity up to additive constant of such coordinates, there is $a(t)$ such that $\Psi_Q^+ \circ T_{a(t)} \circ \zeta_t = \chi_t \circ \Psi_Q^+$. Set $\xi_t = T_{a(t)} \circ \zeta_t$. It is again an integrating map of μ_t^+ and satisfies (5). The unicity of such a map is an easy exercise and is left to the reader.

d) Fix $z \in K_Q$ and $\widetilde{z} \in (\Psi_Q^+)^{-1}(z)$. Now the map $t \mapsto \xi_t(\widetilde{z})$ is analytic on t and equals to $\widetilde{z}$ at $t=0$. On the other hand, $\chi_t(z) \equiv z$ for all t by a), so $\xi_t(\widetilde{z}) \in (\Psi_Q^+)^{-1}(z)$ for all t. But the latter set is discrete. Consequently $\xi_t(\widetilde{z}) \equiv \widetilde{z}$ for all t. Arguing similarly on the accesses of $\partial \widetilde{B}_Q(\infty)$ from $\widetilde{B}_Q(\infty)$, one proves that ξ_t is the identity on the ideal boundary of $\widetilde{B}_Q(\infty)$. The universal covering property of $\Psi_Q^+ : \widetilde{B}_Q(\infty) \to B_Q(\infty)\setminus\{\infty\}$ is easy.

The relations of various maps are illustrated in the following commutative diagrams:



□

However, the Lavaurs map $g_{\tilde{\sigma}}$ behaves badly under the above Branner-Hubbard motion. In fact the conjugated map $\chi_t \circ g_{\tilde{\sigma}} \circ \chi_t^{-1}|_{B(\beta')} = \chi_t \circ g_{\tilde{\sigma}}|_{B(\beta')}$ is no more analytic in z. For this reason, following Douady and Lavaurs we introduce a new Q-invariant Beltrami form which will be also $g_{\tilde{\sigma}}$-invariant, as follows: define

$$B_\sigma(\infty) := B_Q(\infty) \cup \{z \mid \exists n \;\; g_{\tilde{\sigma}}^n(z) \in B_Q(\infty)\}.$$

It depends only on the class σ of $\tilde{\sigma}$ in $\mathbb{C}/\mathbb{Z}$. Set

$$\mu_{Q,\sigma} := \begin{cases} (g_{\tilde{\sigma}}^n)^* \mu_Q = (\log \circ \varphi_Q \circ g_{\tilde{\sigma}}^n)^* \frac{d\bar{z}}{dz} & \text{for } z \in B_\sigma(\infty) \text{ and for } n \\ & \text{such that } g_{\tilde{\sigma}}^n(z) \in B_Q(\infty); \\ 0 & \text{for } z \notin B_\sigma(\infty). \end{cases}$$

Fix now $t \in \mathbb{D}$. Note that $t\mu_{Q,\sigma} = (g_{\tilde{\sigma}}^n)^*(t\mu_Q)$, $g_{\tilde{\sigma}}^*(t\mu_{Q,\sigma}) = t\mu_{Q,\sigma}$, $Q^*(t\mu_{Q,\sigma}) = t\mu_{Q,\sigma}$ and $t\mu_{Q,\sigma}$ depends only on the class σ of $\tilde{\sigma}$. We use $\chi_{t,\sigma}$ to denote the integrating map of $t\mu_{Q,\sigma}$, uniquely normalized so that the conjugated map, $Q_{t,\sigma} := \chi_{t,\sigma} Q \chi_{t,\sigma}^{-1}$ is again a monic centered polynomial. Clearly $Q_{t,\sigma}$ has again a connected Julia set, and a parabolic fixed point $\beta'_{t,\sigma} := \chi_{t,\sigma}(\beta')$ of multiplier 1, with $B_{t,\sigma} := \chi_{t,\sigma}(B(\beta'))$ as an immediate basin. We may thus define the corresponding objects in the list (4) for $Q_{t,\sigma}$, in particular the corresponding Böttcher/Fatou coordinates $\varphi_{t,\sigma}$, $\psi_{t,\sigma}$, $\Phi_{t,\sigma}^\pm$, $\Psi_{t,\sigma}^+$. We choose the normalization so that $\varphi_{t,\sigma}$ is again tangent to the identity at ∞, $\Phi^-_{t,\sigma}(\chi_{t,\sigma}(w_0)) = 0$ and $\Psi^+_{t,\sigma}(0) = \psi_{t,\sigma}(e^x)$, with $x > 0$ independent of t, σ.

Denote by τ_0 the standard complex structure. These maps are related as indicated in the following diagram:

$$
\begin{array}{lcccccc}
Q \hookrightarrow & t\mu_{Q,\sigma}, \mathbb{C} & \xrightarrow{\langle g_{\tilde\sigma}, Q\rangle} & t\mu_Q, \mathbb{C} \supset B_Q(\infty) & \xrightarrow{\varphi_Q} & \mathbb{C}\setminus\overline{\mathbb{D}} \\
 & \chi_{t,\sigma}\big\downarrow & & \big\downarrow\chi_t & & \big\downarrow l_t : z \mapsto z\cdot|z|^{\frac{2t}{1-t}} \\
Q_{t,\sigma} \hookrightarrow & \tau_0, \mathbb{C} & & \tau_0, \mathbb{C} & \xrightarrow[\varphi_Q]{} & \mathbb{C}\setminus\overline{\mathbb{D}}
\end{array}
$$

Denote by $\mu^+_{t,\sigma} := (\Psi^+_Q)^*(t\mu_{Q,\sigma})$ and $\mu^-_{t,\sigma} := T^*_{\tilde\sigma}\mu^+_{t,\sigma}$.

Lemma 4.2. *Fix $t \in \mathbb{D}$ and $\sigma \in \mathbb{C}/\mathbb{Z}$. The objects of $Q_{t,\sigma}$ and those of Q are related as follows:*

a) $\psi_{t,\sigma} = \chi_{t,\sigma} \circ \psi_Q \circ l_t^{-1}$ *on* $\mathbb{C}\setminus\overline{\mathbb{D}}$*;*

b) *For $\eta^-_{t,\sigma}$ the integrating map of $\mu^-_{t,\sigma}$ fixing* $0, 1$ *and* ∞*, we have* $\Phi^-_{t,\sigma} = \eta^-_{t,\sigma} \circ \Phi^-_Q \circ \chi^{-1}_{t,\sigma}$*;*

c) *There is a unique integrating* map η^+_t *of $\mu^+_{t,\sigma}$, commuting with T_1 and satisfying* $\Psi^+_{t,\sigma} = \chi_{t,\sigma} \circ \Psi^+_Q \circ (\eta^+_{t,\sigma})^{-1}$*;*

d) *the* map $\eta^+_{t,\sigma} \circ T_{\tilde\sigma} \circ (\eta^-_{t,\sigma})^{-1}$ *is a translation and $\chi_{t,\sigma} g_{\tilde\sigma} \chi^{-1}_{t,\sigma}$ is a Lavaurs map of* $Q_{t,\sigma}$.

e) *If, for some t, σ, we have $Q_{t,\sigma} = Q$, then $\chi_{t,\sigma} = \chi_t$ on $B_Q(\infty)$ and $\chi_{t,\sigma} = id$ on the ideal boundary of $B_Q(\infty)$.*

f) *If $Q_{t,\sigma} = Q$ for all $t \in \mathbb{D}$, then $\eta^+_{t,\sigma}$ coincides with ξ_t of Lemma* 4.1.c) *on $\widetilde{B}_Q(\infty)$ and in particular $\eta^+_{t,\sigma} = id$ on the ideal boundary of $\widetilde{B}_Q(\infty)$.*

Proof. The proof is very similar to that of Lemma 4.1. We give at first a sketch in the following commutative diagram.

$$
\begin{array}{ccccccc}
\mathbb{C}^-, 0 & \xleftarrow{\eta^-_{t,\sigma}} & \mathbb{C}^-, 0 & \xrightarrow{T_{\tilde\sigma}} & \mathbb{C}^+, 0 & \xrightarrow{\eta^+_{t,\sigma}} & \mathbb{C}^+, b(t) \\
\Phi^-_{t,\sigma}\big\uparrow & & \Phi^-_Q\big\uparrow & & \Psi^+_Q\big\downarrow & & \big\downarrow\Psi^+_{t,\sigma} \\
B_{t,\sigma}, \chi_{t,\sigma}(w_0) & \xleftarrow[\chi_{t,\sigma}]{} & B(\beta'), w_0 & \xrightarrow[g_{\tilde\sigma}]{} & \mathbb{C}, \psi_Q(e^x) & \xrightarrow[\chi_{t,\sigma}]{} & \mathbb{C}, \psi_{t,\sigma}(e^{\frac{1+t}{1-t}x}) \\
 & & & & \psi_Q\big\uparrow & & \big\uparrow\psi_{t,\sigma} \\
 & & & & \mathbb{C}\setminus\overline{\mathbb{D}}, e^x & \xrightarrow[l_t]{} & \mathbb{C}\setminus\overline{\mathbb{D}}, e^{\frac{1+t}{1-t}x}
\end{array}
\tag{6}
$$

In this diagram, $x > 0$ is the real number such that $\Psi^+_Q(0) = \psi_Q(e^x)$; the maps $g_{\tilde\sigma}$ and $T_{\tilde\sigma}$ do not necessarily preserve the base points, whereas the other maps do.

a) One checks easily that $\chi_{t,\sigma} \circ \psi_Q \circ l_t^{-1}$ is conformal, conjugates z^d to $Q_{t,\sigma}$, depends continuously on t, is tangent to the identity at ∞ when $t = 0$, and tangent to a $(d-1)$th-root of unity at ∞ for any t. One concludes then for every t, the map $\chi_{t,\sigma} \circ \psi_Q \circ l_t^{-1}$ is tangent to the identity at ∞, coincides thus with $\psi_{t,\sigma}$.

b) and c). As in the proof of the previous lemma, the structure $\mu^+_{t,\sigma}$ is T_1-invariant. So $\mu^-_{t,\sigma}$ is $T_{\tilde\sigma}^{-1} T_1 T_{\tilde\sigma}$-invariant, thus is T_1-invariant. Arguing as for ζ_t, ξ_t in the previous Lemma, one concludes that $\eta^-_{t,\sigma} \circ \Phi^-_Q \circ \chi^{-1}_{t,\sigma}$ coincides with the attracting Fatou coordinates of $Q_{t,\sigma}$, whereas $\chi_{t,\sigma} \circ \Psi^+_Q \circ (\eta^+_{t,\sigma})^{-1}$ coincides with the repelling Fatou coordinates of $Q_{t,\sigma}$, after suitable normalization of the integrating map $\eta^+_{t,\sigma}$.

d) The map $\eta^+_{t,\sigma} \circ T_{\tilde\sigma} \circ (\eta^-_{t,\sigma})^{-1}$ is a conformal automorphism of $\mathbb{C}$, as $T^*_{\tilde\sigma}\mu^+_{t,\sigma} = \mu^-_{t,\sigma}$. It is therefore of the form $az + c$. But it commutes with T_1, so it is of the form $z + c$, i.e., a translation. It follows by definition that $\chi_{t,\sigma} g_{\tilde\sigma} \chi^{-1}_{t,\sigma}$ is a Lavaurs map of $Q_{t,\sigma}$.

e) Assume that for some given t, σ we have $Q_{t,\sigma} = Q$. This implies $B_{t,\sigma} = B(\beta')$, $\psi_{t,\sigma} = \psi_Q$, $\Psi^+_{t,\sigma} = \Psi^+_Q$ and $\Phi^-_{t,\sigma} = \Phi^-_Q$. So the diagram (6) becomes

$$
\begin{array}{ccccccc}
\mathbb{C}^-, 0 & \xleftarrow{\eta^-_{t,\sigma}} & \mathbb{C}^-, 0 & \xrightarrow{T_{\tilde\sigma}} & \mathbb{C}^+, 0 & \xrightarrow{\eta^+_{t,\sigma}} & \mathbb{C}^+, b(t) \\
\uparrow{\scriptstyle \Phi^-_Q} & & \uparrow{\scriptstyle \Phi^-_Q} & & \downarrow{\scriptstyle \Psi^+_Q} & & \downarrow{\scriptstyle \Psi^+_Q} \\
B(\beta'), w_0 & \xleftarrow[\chi_{t,\sigma}]{} & B(\beta'), w_0 & \xrightarrow[g_{\tilde\sigma}]{} & \mathbb{C}, \psi_Q(e^x) & \xrightarrow[\chi_{t,\sigma}]{} & \mathbb{C}, \psi_Q(e^{\frac{1+t}{1-t}x}) \\
 & & & & \uparrow{\scriptstyle \psi_Q} & & \uparrow{\scriptstyle \psi_Q} \\
 & & & & \mathbb{C}\setminus\overline{\mathbb{D}}, e^x & \xrightarrow[l_t]{} & \mathbb{C}\setminus\overline{\mathbb{D}}, e^{\frac{1+t}{1-t}x}
\end{array}
\tag{7}
$$

It follows from a) that $\chi_{t,\sigma} = \psi_Q \circ l_t \circ \psi_Q^{-1}$ on $B_Q(\infty)$. By Lemma 4.1.a), $\chi_{t,\sigma}$ coincides with χ_t on $B_Q(\infty)$ and is the identity on the ideal boundary.

f) Arguing as in the proof of Lemma 4.1.d), one proves that $\eta^+_{t,\sigma}$ is the identity on the ideal boundary of $\widetilde{B}_Q(\infty)$. Now $\eta^+_{t,\sigma} \circ \xi_t^{-1}$ on $\widetilde{B}_Q(\infty)$ is a conformal automorphism (as they integrate the same complex structure) and is the identity on the ideal boundary. It follows that $\xi_t = \eta^+_{t,\sigma}$ on $\widetilde{B}_Q(\infty)$. □

Although χ_t leaves Q invariant, $\chi_{t,\sigma}$ may deform Q to nearby maps. Even in the case $\chi_{t,\sigma}$ leaves Q invariant, it often deforms $g_{\tilde\sigma}$ to other Lavaurs maps (see below).

4.2. Fatou vectors for $Q \in \mathcal{A}$. In this subsection, fix $Q \in \mathcal{A}$, i.e a cubic cauliflower. Denote as before by β' the unique parabolic fixed point of multiplier 1, and by $B(\beta')$ the unique immediate basin of β'. The following is classical:

Lemma 4.3. *For any polynomial P close to Q, there are exactly two fixed points α_1, α_2 (counting with multiplicity) of P close to β'. For $j = 1, 2$ denote by λ_j the multiplier of P at α_j and $\tilde\sigma_j = \frac{2\pi i}{1-\lambda_j}$. Assume that both α_1 and α_2 are repelling. Then there are two constants $C \in \mathbb{C}$ and $C' \in \mathbb{R}^-$ depending on Q such that*

a) *$\tilde{\sigma}_1 + \tilde{\sigma}_2 \to C$ as $P \to Q$;*
b) *$|\tilde{\sigma}_i| \to \infty$, as $P \to Q$, for $i = 1, 2$;*
c) *$C' < \Im(\tilde{\sigma}_i) < \pi$, for $i = 1, 2$;*
d) *exchanging the labeling of α_1, α_2 if necessary, $\Re(\tilde{\sigma}_1) \to -\infty$ as $P \to Q$.*

Proof. a) is due to the continuity of holomorphic indices. See e.g. [Mi].
b) is due to the fact that $\lambda_i \to 1$ as $P \to Q$.
c) By assumption $|\lambda_i| > 1$, i.e., $\lambda_i \in \overline{\mathbb{C}} \backslash \overline{\mathbb{D}}$. The Möbius map $h(w) = \frac{2\pi i}{1-w}$ maps $\overline{\mathbb{C}} \backslash \overline{\mathbb{D}}$ onto $\{\Im(w) < \pi\}$. Therefore $\Im(\tilde{\sigma}_i) = \Im h(\lambda_i) < \pi$. On the other hand, by a), $\Im(\tilde{\sigma}_1) + \Im(\tilde{\sigma}_2)$ remains bounded. So each $\Im(\tilde{\sigma}_i)$ is bounded from below.
d) Combining a), b) and c), we conclude that $\Re(\tilde{\sigma}_1) + \Re(\tilde{\sigma}_2)$ remains bounded and $|\Re(\tilde{\sigma}_i)| \to \infty$, for $i = 1, 2$. Therefore one of $\Re(\tilde{\sigma}_1)$, $\Re(\tilde{\sigma}_2)$ tends to $-\infty$ as $P \to Q$. □

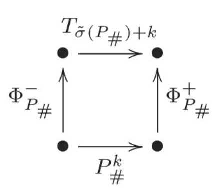
Definition. For $P_\#$ a perturbed map of Q without attracting fixed points, we define **the lifted phase** $\tilde{\sigma}(P_\#)$ to be one of the $\tilde{\sigma}_i$ with large negative real part, as indicated in Lemma 4.3. Denote by $\sigma(P_\#)$ its class in $\mathbb{C}/\mathbb{Z}$.

Corollary 4.4. *Assume P_n is a sequence of cubic polynomials without attracting fixed points, and converging algebraically to Q. Then, taking a subsequence if necessary, we have $\sigma(P_n) \to \sigma$ in $\mathbb{C}/\mathbb{Z}$.*

Proof. By Lemma 4.3 we have $\tilde{\sigma}(P_n) = a_n + ib_n$ with $C' < b_n < \pi$ and $a_n \to -\infty$. Denote by $\{a_n\}$ the fractional part of a_n. Taking a subsequence if necessary we have $\{a_n\} \to a \in [0, 1[$ and $b_n \to b \in [C', \pi]$. Consequently $\sigma(P_n)$ converges to the class of $a + ib$ in $\mathbb{C}/\mathbb{Z}$. □

The following is due to Lavaurs Douady:

Lemma 4.5. *Another meaning of $\tilde{\sigma}(P_\#)$ is as follows: there exist Fatou coordinates $\Phi^{\pm}_{P_\#}$ for $P_\#$, and they can be normalized suitably so that they depend continuously on $P_\#$ and that $\Phi^{+}_{P_\#}(z) - \Phi^{-}_{P_\#}(z) \equiv \tilde{\sigma}(P_\#)$. In other words, for suitably chosen z, and some N depending on $P_\#$ (tending to $+\infty$ as $P_\# \to Q$), we have $P^k_\#(z) = (\Phi^{+}_{P_\#})^{-1} \circ T_{\tilde{\sigma}(P_\#)+k} \circ \Phi^{-}_{P_\#}(z)$ for $k = 0, 1, \ldots, N$.*



In case $P_n \to Q$ and $\sigma(P_n) \to \sigma$, for $\tilde{\sigma} \in \mathbb{C}$ a lift of σ and $k_n \in \mathbb{N}$ such that $\tilde{\sigma}(P_n) + k_n \to \tilde{\sigma}$, we have $T_{\tilde{\sigma}(P_n)+k_n} \to T_{\tilde{\sigma}}$, $\Phi^{\pm}_{P_n} \to \Phi^{\pm}_{Q}$, and therefore $P^{k_n}_n \to g_{\tilde{\sigma}}$ uniformly on compact sets of K_Q. Based on this, we have

Theorem 4.6 (Key continuity theorem, [W, 8.2]). *Denote by $K(Q,\sigma)$, $J(Q,\sigma)$ the enriched filled Julia set and the enriched Julia set. More precisely $K(Q,\sigma)=K_Q\backslash B_\sigma(\infty)$ and $J(Q,\sigma)=\partial K(Q,\sigma)$ (see [D2] for more details). Assume*

C0. $(Q,\sigma)\in\mathcal{A}\times\mathbb{C}/\mathbb{Z}$ *and $g_{\tilde\sigma}$ sends the two critical points of Q outside K_Q.*
A. $P_n\to Q\in\mathcal{A}$, *and* $P_n\notin\mathcal{A}$.
B. $\sigma(P_n)\to\sigma\in\mathbb{C}/\mathbb{Z}$.

These conditions imply C1. $K_{P_n}\to K(Q,\sigma)$, $J_{P_n}\to J(Q,\sigma)$ *and* $mes(J(Q,\sigma))=0$.
Assume furthermore D. $t_n\to t_0\in\mathbb{D}$. *Then*
A, B, $C1$ *and* $D\Longrightarrow$
E. $t_n\mu_{P_n}\longrightarrow_{a.e.}t_0\mu_{Q,\sigma}$; $\Longrightarrow$
F. $\chi_{t_n,P_n}\to\chi_{t_0,\sigma}$ *uniformly on* $\overline{\mathbb{C}}$; $\Longrightarrow$
G. $S(t_n,P_n)\to Q_{t_0,\sigma}$.

Sketch of a proof: Part $A+B+C0\Longrightarrow C1$ is a theorem of Douady-Lavaurs. Part $E\Longrightarrow F$ can be found in [L, Theorem 4.6]. The remaining part should be checked by hand. □

Definition (three half neighborhoods of β' and $-\infty\in\mathbb{C}^+$).

- Define $V_Q=\Psi_Q^+(\{\Re w<0\})$. In other words Φ_Q^+ is well defined and univalent on V_Q, mapping it onto the left half plane.
- Define $L_Q=\psi_Q(e^R)$, where $R=\{s+i\theta\ |0<s<s_0,-\theta_0<\theta<\theta_0\}$ is a small rectangle so that $L_Q\subset V_Q$.
- Define $M<0$ large enough so that

$$\{\Re w<M\}\cap\widetilde{B}_Q(\infty)\subset\Phi_Q^+(L_Q)\subset\{\Re w<0\},\quad\text{and}$$
$$\Psi_Q^+(\{\Re w<M\})\cap B_Q(\infty)\subset L_Q\subset V_Q.$$

Definition. Define the **Fatou vector** $v(Q)$ to be $\Phi_Q^-(w_1)-\Phi_Q^-(w_0)$, the difference of the two critical points in the attracting Fatou coordinates. It depends on the labelling of the critical points, but not on the normalization of Φ_Q^-.

4.3. Discontinuity of the wring operator.

Proposition 4.7.I ([W, 8.12]). *There are $Q\in\mathcal{A}$ such that, for any given labeling w', w'' of the critical points of Q, there is $\tilde\sigma\in\mathbb{C}$, such that, for L_Q defined as above, we have $T_{\tilde\sigma}\circ\Phi_Q^-(w')$, $T_{\tilde\sigma}\circ\Phi_Q^-(w'')\in\Phi_Q^+(L_Q)$, $g_{\tilde\sigma}(w')\in L_Q\cap R_Q(0)$ and $g_{\tilde\sigma}(w'')\in L_Q\backslash R_Q(0)$.*

For any such couple $(Q,\tilde\sigma)$, and for σ the class in $\mathbb{C}/\mathbb{Z}$ of $\tilde\sigma$,

II. *σ satisfies C0 and $t\mapsto Q_{t,\sigma}$ is non constant (a sufficient condition for this is that the ground wind $w(Q,\sigma)$ does not vanish,* [W, 8.12]).

III. *$(t, P) \mapsto S(t, P)$ is discontinuous at (t, Q) for any $t \in \mathbb{D}^*$ with $Q_{t,\sigma} \neq Q$* ([W, 8.5, 8.9]).

Proof. I. In the quotient repelling Ecalle cylinder, the quotient $[R_Q(0)]$ of the 0-ray is the core curve of the annulus $[B_Q(\infty)]$, of modulus $\frac{\pi}{\log 3}$ (independent of Q, this can be seen in the Böttcher coordinate). For any real polynomial $Q \in \mathcal{A}$ with $v(Q) \neq 0$, the map Φ_Q^+ is real, $R_Q(0) \subset \mathbb{R}$ and K_Q is symmetric with respect to $\mathbb{R}$. There is a universal constant h_0 such that Ψ_Q^+ maps the strip $\{u + iy | \; |y| \leq h_0\}$ into $B_Q(\infty)$. Note that Ψ_Q^+ maps $\mathbb{R}$ onto $R_Q(0)$ and maps $\mathbb{R}^-$ onto $R_Q(0) \cap V_Q$. We may then choose a real Q such that $|\Im v(Q)| \leq h_0$ and $\Im v(Q) \neq 0$ (such map exists, see §5). Define the three half neighborhood of β' of Q as in §4.2. We have a fourth half neighborhood consisting of $\Psi_Q^+(\{u + iy | u < M, |y| < h_0\})$, which is contained in $B_Q(\infty) \cap L_Q \subset V_Q$.

Fix a labeling w', w'' of the critical points of Q.

Now choose a translation $T_{\tilde{\sigma}}$ so that $a := T_{\tilde{\sigma}}(\Phi_Q^-(w'))$ is a large negative real number satisfying

$$\max\{a, a \pm \Re v(Q)\} < M.$$

Then

$$\begin{aligned} T_{\tilde{\sigma}}(\Phi_Q^-(w'')) &= T_{\tilde{\sigma}}(\Phi_Q^-(w') \pm v(Q)) \\ &= T_{\tilde{\sigma}}(\Phi_Q^-(w')) \pm v(Q) = a \pm \Re v(Q) \pm i \Im v(Q), \end{aligned}$$

So both $g_{\tilde{\sigma}}(w') = \Psi_Q^+ \circ T_{\tilde{\sigma}} \circ \Phi_Q^-(w')$ and $g_{\tilde{\sigma}}(w'') = \Psi_Q^+ \circ T_{\tilde{\sigma}} \circ \Phi_Q^-(w'')$ are contained in the fourth half neighborhood of β', satisfying in particular $g_{\tilde{\sigma}}(w') \in L_Q \cap R_Q(0)$ and $g_{\tilde{\sigma}}(w'') \in L_Q \setminus R_Q(0)$.

II. Clearly σ satisfies C0, that is, $g_{\tilde{\sigma}}$ maps the two critical points of Q outside K_Q.

Assume by contradiction that $Q_{t,\sigma} \equiv Q$ for all $t \in \mathbb{D}$. Then $\chi_{t,\sigma}$ commutes with Q, in particular $\chi_{t,\sigma}$ maps critical points of Q to critical points of Q. But $t \mapsto \chi_{t,\sigma}(z)$ is analytic on t and is equal to z at $t = 0$. We conclude that $\chi_{t,\sigma}$ is the identity on the grand orbit of the critical points. On the other hand, by Lemma 4.2.e) and f),

$$\begin{aligned} &\chi_{t,\sigma} = \chi_t = \psi_Q \circ l_t \circ \psi_Q^{-1} \quad \text{on} \quad B_Q(\infty), \quad \text{and} \\ &\eta_{t,\sigma}^+ = \xi_t \quad \text{on} \quad \widetilde{B}_Q(\infty). \end{aligned} \tag{8}$$

Fix $-1 < t < 0$. By Lemma 4.2.d) the map $\eta_{t,\sigma}^+ \circ T_{\tilde{\sigma}} \circ (\eta_{t,\sigma}^-)^{-1}$ is some translation T_c. So

$$\begin{aligned} \pm v(Q) = \Phi_Q^-(w') - \Phi_Q^-(w'') &= T_c(\Phi_Q^-(w')) - T_c(\Phi_Q^-(w'')) \\ &= \eta_{t,\sigma}^+ \circ T_{\tilde{\sigma}} \circ (\eta_{t,\sigma}^-)^{-1}(\Phi_Q^-(w')) \\ &\quad - \eta_{t,\sigma}^+ \circ T_{\tilde{\sigma}} \circ (\eta_{t,\sigma}^-)^{-1}(\Phi_Q^-(w'')) \\ &= \eta_{t,\sigma}^+ \circ T_{\tilde{\sigma}}(\Phi_Q^-(w')) - \eta_{t,\sigma}^+ \circ T_{\tilde{\sigma}}(\Phi_Q^-(w'')) \\ &= \xi_t \circ T_{\tilde{\sigma}}(\Phi_Q^-(w')) - \xi_t \circ T_{\tilde{\sigma}}(\Phi_Q^-(w'')) \\ &= \Phi_Q^+ \circ \chi_t \circ g_{\tilde{\sigma}}(w') - \Phi_Q^+ \circ \chi_t \circ g_{\tilde{\sigma}}(w''), \end{aligned}$$

where the second equality is due to the fact that T_c is a translation, the third is due to the previous paragraph, the fourth is due to the fact that $\chi_{t,\sigma}^{-1}$ is the identity on the critical points, the fifth is due to ∞, and finally the last is due to the facts that $g_{\tilde{\sigma}}(w')$, $g_{\tilde{\sigma}}(w'') \in L_Q \subset V_Q$ (by the construction of $\tilde{\sigma}$ and L_Q, V_Q) and that L_Q is invariant by χ_t, $-1 < t < 0$ (by the choice of L_Q).

We claim, however, that the right hand side tends to ∞ as $-1 < t < 0$, $t \searrow -1$. Because, one of the points $\chi_t \circ g_{\tilde{\sigma}}(w')$, $\chi_t \circ g_{\tilde{\sigma}}(w'')$, say the first one, tends to the landing point β' of $R_Q(0)$ whereas the other tends to the landing point γ_θ of $R_Q(\theta)$ for some $\theta \neq 0$. So $\lim_{t\searrow -1} \Phi_Q^+(\chi_t \circ g_{\tilde{\sigma}}(w')) = -\infty$ whereas $\lim_{t\searrow -1} \Phi_Q^+(\chi_t \circ g_{\tilde{\sigma}}(w'')) = \Phi_Q^+(\gamma_\theta) \in \mathbb{C}$.

This is a contradiction.

III. Fix $t \in \mathbb{D}$ such that $Q_{t,\sigma} \neq Q$. Let R' be a compact set contained in the rectangle R such that

$$g_{\tilde{\sigma}}(w'),\, g_{\tilde{\sigma}}(w'') \in \psi_Q(e^{R'}) \subset \psi_Q(e^R) = L_Q.$$

By continuous dependence of $\psi_P|_{R'}$ on P and continuous dependence of $\Phi_P^{\pm}$, one concludes that there is a sequence $P_n \to Q$, $k_n \in \mathbb{N}$ such that $P_n^{k_n}(w_n'),\, P_n^{k_n}(w_n'') \in \psi_{P_n}(e^{R'})$. Therefore J_{P_n} is a Cantor set and all periodic points are repelling. We may thus apply Corollary 4.4 to conclude that $\sigma(P_n) \to \sigma$ (taking a subsequence if necessary). Set $t_n \equiv t \in \mathbb{D}^*$. Then the conditions A, B, C0, D of Theorem 4.6 are satisfied. So $S(t, P_n) \to Q_{t,\sigma} \neq Q \equiv S(t, Q)$, where the last equality is due to Proposition 2.1.(6). Therefore the wring operator $S(t, P)$ is not continuous at (t, Q). □

This proves in particular the part of Theorem 1.2 about the discontinuity of the wring operator on the cubic polynomials.

An interesting necessary condition for a S-ray $S(P)$ to land at a $Q \in \mathcal{A}$ is the following: Assume $S(t, P) := P_t$ converges to $Q \in \mathcal{A}$ as $t \searrow -1$ (with $P \neq Q$). By Theorem 1.1.III, the ray $R_{P_t}(0)$ branches at a critical point denoted by w_t. These points w_t have a limit w as $t \searrow -1$, which is necessarily a critical point of Q. Define

$$\Sigma(w) = \{\sigma \in \mathbb{C}/\mathbb{Z} \mid \exists \tilde{\sigma},\ g_{\tilde{\sigma}}(w) \in R_Q(0) \cap \psi_Q([e^s, e^{3s}])\} \text{ and}$$

$$\Sigma^t = \{\sigma \in \mathbb{C}/\mathbb{Z} \mid T_\sigma([w_t]) \in [R_{P_t}(0) \cap \psi_{P_t}([e^s, e^{3s}])]\}.$$

They are two closed loops in $\mathbb{C}/\mathbb{Z}$.

Lemma 4.8 ([W, 8.7, 8.8]). *In the above setting (landing of an S-ray), assume furthermore $\sigma(P_t) \in \Sigma^t$. Then $d(\sigma(P_t), \Sigma(w)) \to 0$ as $t \searrow -1$. Moreover, if a $\sigma \in \Sigma(w)$ satisfies C0 (i.e., mapping both critical points outside K_Q) then $Q_{t,\sigma} \equiv Q$.*

Proof. By continuity of all involved maps, points, one sees easily that $\Sigma^t \to \Sigma(w)$ in the Hausdorff topology. But $\sigma(P_t) \in \Sigma^t$ by assumption. So $d(\sigma(P_t), \Sigma(w)) \to 0$.

Since the real part of $\tilde{\sigma}(P_t)$ tends to $-\infty$ as $t \searrow -1$, the phase map $t \mapsto \sigma(P_t)$ spirals asymptotically to the loop $\Sigma(w)$. As a consequence, for any $\sigma \in \Sigma(w)$, there is a sequence $t_n \searrow -1$ such that $\sigma(P_{t_n}) \to \sigma$.

Assume a $\sigma \in \Sigma(w)$ satisfies C0. Then by Theorem 4.6, for $-1 < t < 1$,

$$Q \overset{n\to\infty}{\longleftarrow} S_{t\star t_n}(P) = S_t(P_{t_n}) \overset{n\to\infty}{\longrightarrow} Q_{t,\sigma},$$

where $t \star t'$ denotes the group structure in $\mathbb{D}$ related to the operator S. Therefore $Q_{t,\sigma}$ is constant for t real. But it is analytic on t (cf. e.g. [PT, Theorem 2.7]). So it is also constant for all $t \in \mathbb{D}$. $\square$

As for the continuity part of Theorem 1.2, we may apply the following Lemma to the quadratic family (this proof is somewhat different from the two proofs of Willumsen).

Lemma 4.9 ([BH1, 7.2] + [PT]). *For $\mathcal{F}$ an analytic family of polynomials, if S maps $\mathbb{D} \times \mathcal{F}$ into $\mathcal{F}$, then S is a parameter holomorphic motion. It is continuous if in addition $\mathcal{F} \subset \overline{\mathbb{C}}$ (by Słodkovski's theorem).*

Proposition 4.10 ([W, 8.12]). *Assume $S(P) \ni P_n \to Q \in \mathcal{A}$, $P \neq Q$, $\sigma(P_n) \to \sigma$, and $g_{\tilde\sigma}(w) \in B_Q(\infty)$ for a lift $\tilde\sigma$ of σ and for w a critical point of Q. Then, for any lift $\tilde\sigma$ of σ, we have $g_{\tilde\sigma}(w) \in R_Q(0)$ and $P_n^l(w(P_n)) \in R_{P_n}(0)$ for some l independent of n.*

A consequence of this is (improved [W, 8.10]): Assume $(Q, \sigma) \in \mathcal{A} \times \mathrm{C/Z}$ such that for one critical point w' of Q, we have $g_{\tilde\sigma}(w') \in R_Q(\theta')$, with $\theta' \neq 0$. Assume furthermore $P_n \to Q, \sigma(P_n) \to \sigma$. Then $\{P_n\}$ can not belong to a single S-ray.

Proof. All periodic points of P_n are repelling, by Theorem 1.1.I. So we may apply Corollary 4.4 to conclude that there is a subsequence such that $\sigma(P_n)$ converges in C/Z to some σ. Fix a large integer l so that $Q^l(w)$ is in an attracting petal, where Φ_Q^- is injective.

Assume $g_{\tilde\sigma}(w) \in B_Q(\infty)$ (this is independent of the lift). Choose $\tilde\sigma \in \mathrm{C}$ a lift of σ such that $T_{\tilde\sigma+l}(\Phi_Q^-(w))$ has a negative real part. There is therefore a θ such that $T_{\tilde\sigma+l}(\Phi_Q^-(w)) \in \Phi_Q^+(R_Q(\theta))$.

There are integers $k_n \to +\infty$ such that $\tilde\sigma(P_n) + k_n \to \tilde\sigma$ and that $P_n^{k_n}(P_n^l(w(P_n))) \to g_{\tilde\sigma}(Q^l(w))$ (cf. [D2, 18.2]). By semi-continuity of K_{P_n} on n one concludes that $P_n^{k_n}(P_n^l(w(P_n))) \notin K_{P_n}$, and is therefore contained in some $\theta(n)$-ray. Hence the $\frac{\theta(n)}{3^{k_n}}$-ray contains $P_n^l(w(P_n))$ (cf. [Ta, § 2], beware that the σ there is different from the σ here). But $P_n \in S(P)$. By Proposition 3.(2), applied to $z = P_n^l(w(P_n))$, we have $\arg_{P_n}(P_n^l(w(P_n))) = \arg_P(P^l(w(P)))$ and is independent of n. We conclude that $\theta(n) = 0$ for any n. But $\theta(n) \to \theta$. So $\theta = 0$.

From $T_{\tilde\sigma+l}(\Phi_Q^-(w)) \in \Phi_Q^+(R_Q(0))$ on concludes easily that $g_{\tilde\sigma+N}(w) = \Psi_Q^+(T_{\tilde\sigma+N}(\Phi_Q^-(w))) \in R_Q(0)$ for any integer $N \in \mathrm{Z}$.

5. Parameter interpretation, comments. The space $\mathrm{Per}_1(1)$ of cubic polynomials with a fixed point of multiplier 1 modulo affine conjugacy can be parametrized by

$b^2 \in \mathbb{C}$ with $\tilde{Q}_b(z) = z^3 + bz^2 + z$, as two such maps are conjugate iff $b = -b'$. Figure 2 represents the b-plane, centered at 0. The set $\mathcal{A}$ in this plane consists of the two large butterfly wings. The locus of parabolic attracting maps in $\mathcal{A}$ is the lemniscate $\{b, |b^2 - \frac{1}{2}| < \frac{1}{2}\}$ (Figure 3). The parameter interpretation of Theorem 1.1.I is that no S-ray accumulates at this lemniscate.

Figure 4 represent the right wing of $\mathcal{A}$ in the b-plane. It is quite easy to check that in this plane $\mathcal{A} \cap \mathbb{R}^+ = \;]0, 2[$. The examples constructed in Proposition 4.7 correspond to $b \in \;]0, \sqrt{3}[$, close to $\sqrt{3}$ (see Figure 4). The conclusion is that no S-rays lands at such b.

Komori and Nakane have proved the following closely related results (see [KN]):

There is a monotonic sequence of points $b_n \in [\sqrt{3}, 2[$, $b_0 = \sqrt{3}$, $b_n \nearrow 2$ with integer Fatou vectors. Each of them is the landing point of some S-ray. The in-between real S-rays accumulate but do not land in $[\sqrt{3}, 2]$.

The paper [PT2] contains the following parameterization of each wing of $\mathcal{A}$:

Denote by B the parabolic basin of $F: z \mapsto z^2 + \frac{1}{4}$, by $\Phi^- : B \to \mathbb{C}$ the attracting Fatou coordinates normalized so that $\Phi^-(0) = 0$. Then $(\Phi^-)^{-1}(\mathbb{R})$ decomposes B into a chess board structure, mapping each open chess square bi-holomorphically onto the upper half or the lower half plane, and mapping the borders onto the real line. Denote by U the component of $(\Phi^-)^{-1}(\mathbb{H}_r)$ containing the parabolic fixed point on the boundary. Define $\sim$ on ∂U by $z \sim \bar{z}$.

There exists a bi-holomorphic map $\mathcal{H}$ from each component of $\mathcal{A}$ onto $B \backslash U / \sim$, recording the position of the second attracted critical point (the quotient is meant to cancel the confusing case when both critical points can be considered as 'first'). One proves easily $v(Q) = \Phi^- \circ \mathcal{H}(Q)$. There is therefore a chess board structure on $\mathcal{A}$ (see Figures 4 and 5). Moreover $\Phi^- \circ \mathcal{H}$ maps each of the two main chess squares (the ones in Figures 4 with $]0, \sqrt{3}[$ as an edge) bi-holomorphically onto the upper-left quarter and the lower-left quarter plane, and maps each of the other chess squares onto the upper half or the lower half plane. This implies in particular the surjectivity of $Q \mapsto v(Q)$, $\mathcal{A} \to \mathbb{C}$.

The results listed in this article lead naturally to the following questions:

1. Are the following conditions equivalent?
   - $\mathrm{Acc}(S(P)) \cap \mathrm{Per}_1(1) \neq \emptyset$,
   - $\mathrm{Acc}(S(P)) \subset \mathrm{Per}_1(1)$, and
   - either $P \in \mathrm{Per}_1(1)$ or a fixed ray of P branches.
2. $\mathrm{Acc}(S(P)) \cap \mathcal{A} \neq \emptyset \Longleftrightarrow \mathrm{Acc}(S(P)) \subset \mathcal{A} \Longleftrightarrow$ the set $\bigcup_n P^{-n}(R_P(0))$ (or $\frac{1}{2}$)) is connected, and necessarily contains both critical points.
3. In the setting of Lemma 3, is the assumption $\sigma(P_t) \in \Sigma^t$ always satisfied?

Comments. This is a free extraction of some of the results in Pia Willumsen's thesis, with a different organization and sometimes a different proof. It contains also some generalizations of the original results. I take the responsibility for all eventual introduced errors. The reader is strongly advised to read the original document, which

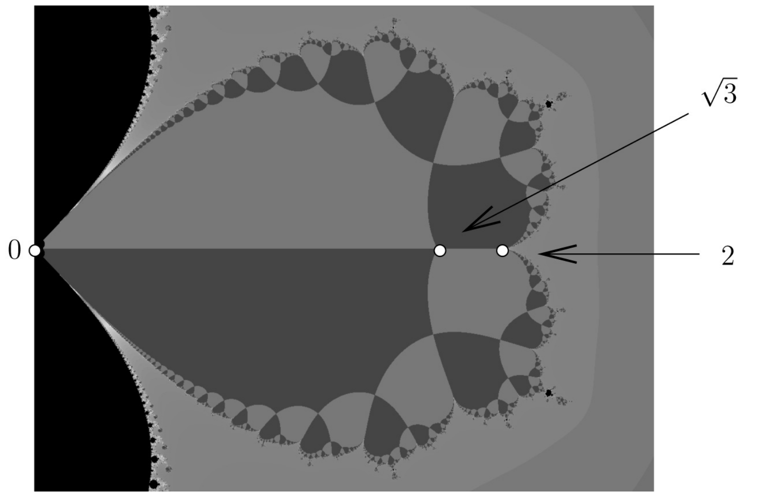


Figure 4: The chess board structure in $\mathcal{A}$

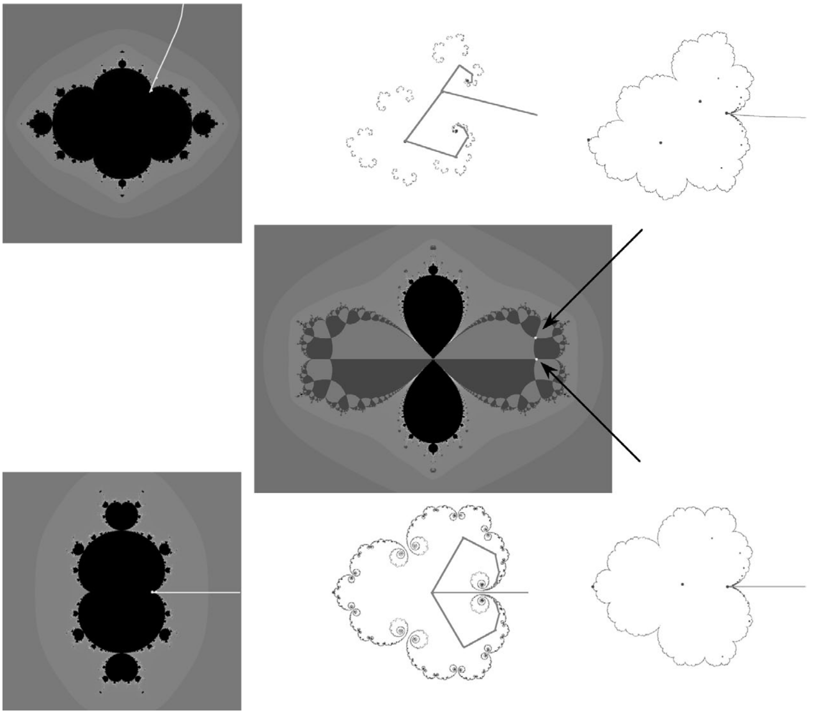


Figure 5: Julia sets for pairs (P, Q), left and right limit 0-rays for P and the 0-ray for Q, and parameter slices containing these maps.

includes a lot more related results, background material, complete proofs as well as many impressive figures.

Acknowledgement. First of all, I would like to thank Pia Willumsen for her long term friendship and for her permission to let me write this article. She also provided me some first hand materials, including the wonderful computer illustrations, drawn using the powerful program of D. Sørensen. Thanks go to as well C. Petersen, P. Roesch and Sh. Nakane for helpful comments.

Main Reference:

[W] P. WILLUMSEN, On accumulation of stretching rays, ph.d. thesis, Department of Mathematics, Technical University of Denmark, February 1997.

References

[BH1] B. BRANNER, & J. H. HUBBARD, The iteration of cubic polynomials? Part I: The global topology of parameter space, Acta Math. 160 (1988), 143–206.

[BH2] B. BRANNER, & J. H. HUBBARD, The iteration of cubic polynomials. Part II: Patterns and parapatterns, Acta Math. 169 (1992), 229–325.

[D2] A. DOUADY, Does a Julia set depend continuously on the polynomial? Proceedings of Symposia in Applied Mathematics 49 (1994), 91–138.

[KN] Y. KOMORI, & SH. NAKANE, Landing property of stretching rays for real cubic polynomials, Conformal geom. and dyn. 8 (2004), 87–114.

[L] O. LEHTO, Univalent Functions and Teichmüller spaces, Springer 109, 1987.

[LP] G. LEVIN, & F. PRZYTYCKI, External rays to periodic points, Israel J. of Math. 94 (1996), 29–57.

[Mi] J. MILNOR, Dynamics in one complex variable: Introductory Lectures, Friedr. Vieweg & Sohn. Braunschweig, 1999.

[Na] V. I. NAISHUL, Topological invariants of analytic and area preserving mappings and their application to analytic differential equations in $\mathbb{C}^2$ and $\mathbb{C}\mathrm{P}^2$, Trans. Moscow Math. Soc. 42 (1983), 239–250.

[P-M] R. PÉREZ-MARCO, Fixed points and circle maps, Acta Math., 179 (1997), 243–294.

[Pe] PETERSEN, C., On the Pommerenke-Levin-Yoccoz inequality, Erg. Th. and Dyn. Sys. 13 (1993), 785–806.

[PR] C. PETERSEN, & G. RYD, Convergence of rational rays in parameter spaces, in The Mandelbrot set, Theme and Variations, LMS Lecture Note Series 274, Cambridge Univ. Press, 2000, 161–172.

[PT] C. PETERSEN, & L. TAN, Branner-Hubbard motions and attracting dynamics, in this volume.

[PT2] C. PETERSEN, & L. TAN, Parabolic slices of cubic polynomials, in preparation.

[ST] M. SHISHIKURA & L. TAN, An alternative proof of *Mañé*'s theorem on non-expanding Julia sets, in The Mandelbrot set, Theme and Variations, LMS Lecture Note Series 274, Cambridge Univ. Press, 2000, 265–279.

[T] L. TAN, Local properties of the Mandelbrot set at parabolic points, in The Mandelbrot set, Theme and Variations, LMS Lecture Note Series 274, Cambridge Univ. Press, 2000, 133–160.

Laboratoire d'Analyse, Géométrie et Modélisation, UMR 8088
Université de Cergy-Pontoise
2 Avenue Adolphe Chauvin
95302 Cergy-Pontoise Cedex, France
e-mail: tanlei@math.u-cergy.fr

Conjectures about the Branner-Hubbard motion of Cantor sets in C

Adrien Douady

Introduction. For $c \in \mathbf{C} - M$, the filled Julia set K_c of $z^2 + c$ is a Cantor set. For almost all external ray $\mathcal{R}$ of M, when c moves on $\mathcal{R}$ and tends to M, the point c has a limit c_0, the set K_{c_0} is a dendrite and K_c tends to K_{c_0}.

The present paper is a reflection on the following question: How much of this fact is really related to dynamics? We only present conjectures, we have no results. This is part of the program "Dynamics without dynamics" (cf [D/V])

Moving in $\mathbf{C} - M$ along external rays (and equipotentials) is a particular case of the Branner-Hubbard motion, which can be defined for any Dirichlet regular compact set K, providing a family $(K_\lambda)_{\lambda \in \mathbf{H}_+}$, where $\mathbf{H}_+ = \{z \in \mathbf{C} \mid Re(z) > 0\}$, with $K_1 = K$.

We conjecture that, for a large class of Cantor sets in $\mathbf{C}$ (which includes the Cantor quadratic Julia sets), the following holds: for almost all β, the Cantor set $K_{\alpha+i\beta}$ has a limit $K_{i\beta}$ when α tends to 0, and this limit is a dendrite. We present also variants of this conjecture in a more extended situation.

The Branner-Hubbard motion has been introduced in [BH2], for cubic Julia sets with one critical point escaping. It has been used by Binder-Makarov-Smirnov [BMS]. We have benefited of talks by M. Zinsmeister on [BMS] in the Seminar COOL (Cergy-Orleans-Orsay-Lille), and of inspiring discussions with him, J.-P. Otal, P. Haïssinsky, P. Sentenac, Tan Lei, M. Flexor, B. Branner, Ch. Henriksen, C. Petersen and others. We thank C. Petersen for many comments, resulting in improvement and clarification of earlier versions.

1. The euclidean harmonic metric. Let $K \subset \mathbf{C}$ be a Dirichlet regular compact set. This means that there is a function $G = G_K : \mathbf{C} \to \mathbf{R}_+$, called the *potential*, satisfying:

(i) G is continuous on $\mathbf{C}$ and vanishes on K:
(ii) G is harmonic on $\mathbf{C} - K$;
(iii) $G(z) = \mathrm{Log}|z| + O(1)$ when z tends to ∞.

In fact, $G(z) = \mathrm{Log}|z| - \mathrm{Log} R + o(1)$ for some R called the *capacity radius* of K. We shall always suppose that K is *monic*, i.e., with capacity radius 1. Then we can always *center* K, i.e., translate it so that $G(z) = \mathrm{Log}|z| + o(\frac{1}{z})$.

The *equipotential* of level s is the set $G^{-1}(s)$. An *equipotential component* is a connected component of an equipotential. A *critical equipotential component* is an equipotential component which contains a critical point of G. The set $\mathbf{C} - (K \cup \bigcup E_i)$, where the E_i are the critical equipotential components, has one unbounded connected component, called the *overcritical annulus*, the other components are the *intercritical annuli*.

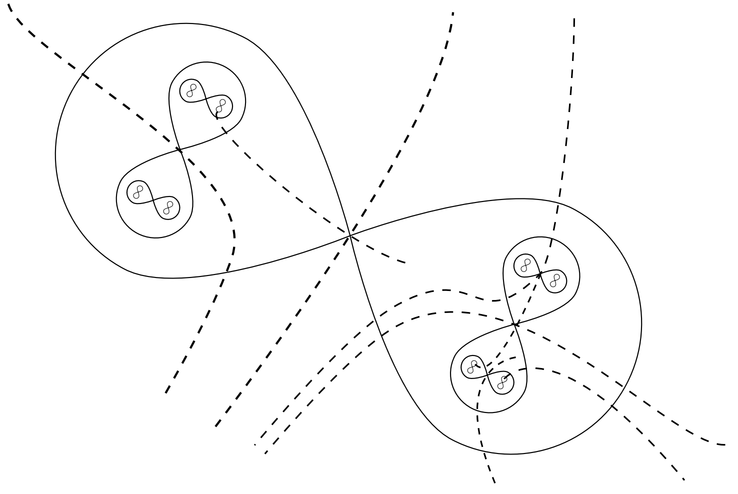


Figure 1: Equipotentials and critical rays.

The *level of separation* of two points x and y in K is the smallest level of an equipotential component which encloses both x and y. If K is a Cantor set, the level of separation is an ultrametric distance on K which defines its topology. The level of separation $G^{sep}(K)$ of K is the level of the highest critical point of G, it is the level of the inner boundary of the overcritical annulus.

A *harmonic chart* is a $\mathbf{C}$-analytic homeomorphism $\phi : U \to \phi(U)$ such that $Re(\phi) = G|_U$, with U an open set in $\mathbf{C} - K$. On a neighbourhood of any point $z \in \mathbf{C} - K$ which is not a critical point of G, there is a harmonic chart defined. The harmonic charts form an atlas on $\Omega = C - (K \cup \mathrm{Crit}(G))$ called the *harmonic atlas*, in which the change of charts are of the form $Z \mapsto Z + ib$, with $b \in \mathbf{R}$ locally constant.

We define the *harmonic euclidean metric* on Ω as the riemannian metric whose expression in the harmonic charts is the standard metric $|dZ|$. The distance defined by this metric extends to a distance on $\mathbf{C} - K$, compatible with its topology. In a neighbourhood of a critical point of G of multiplicity k, the space $\mathbf{C} - K$, provided with this distance, is locally isometric to the space obtained by sewing $2k+2$ half planes.

If K is connected, then $\mathbf{C} - K$ is isometric to $\mathbf{R}_+^* \times \mathbf{R}/2\pi\mathbf{Z}$. In the general case, the overcritical annulus is isometric to $\mathbf{R}_+^* \times \mathbf{R}/2\pi\mathbf{Z}$, each intercritical annulus A is isometric to an open cylinder $]0, h[\times \mathbf{R}/\ell\mathbf{Z}$, with $h = G^+ - G^-$ where G^+ (resp. G^-) is the level of the upper (resp. lower) boundary of A, and $\ell = 2\pi\omega(K_A)$ where ω denotes the harmonic measure and K_A is the part of K enclosed by A.

Still supposing K monic, there is a holomorphic map $\beta = \beta^K : V \to \overline{\mathbf{C}}$ with V a neighbourhood of ∞ in $\overline{\mathbf{C}}$, having ∞ as a fixed point, such that $G = \mathrm{Log}|\beta|$ and $\beta(z) = z + O(1)$. This generalizes the Böttcher coordinate of a polynomial (the harmonic charts generalize the LogBöttcher coordinates). If K is centered, $\beta^K(z) = z + o(1)$.

The *quotient tree* is the quotient T_K of $\overline{\mathbf{C}} - K$ by the equivalence relation contracting each equipotential component to a point. It is a rooted tree (root at infinity); the branch points correspond to critical equipotential components.

2. The quadratic case and the dyadic case. Consider $f_c : z \mapsto z^2 + c$ with $c \in \mathbf{C} - M$, set $K = K_c = K(f_c)$, and $G = G_c = G_{K_c}$. The potential of c with respect to M is

$$G_M(c) = G_c(c) = 2 \cdot G_c(0).$$

Let θ be the external argument of c with respect to M, which is also its argument with respect to K_c.

The quotient tree is a dyadic tree. Each cylinder has 3 marked points: one critical point of G on its upper boundary (except the overcritical cylinder), and two on its lower boundary to be collapsed to one point. The cylinders are labeled by words in the alphabet $\{0, 1\}$; each cylinder C_w has two "children" $C_{w,0}$ and $C_{w,1}$, and f induces a $\mathbf{C}$-analytic isomorphism of C_w onto $C_{\alpha(w)}$ where $\alpha(w)$ is w with the first letter removed. The cylinder $C_\emptyset$ is not the whole overcritical annulus, but the cylinder bounded above by the equipotential passing through the critical value c (which is then a marked point). The cylinders of order k (corresponding to a word with k letters) are mapped onto $C_\emptyset$ by f^k. Such a cylinder has height $2^{-(k+1)} G_M(c)$, circumference $2^{-k} \cdot 2\pi$, thus modulus $\frac{G_M(c)}{4\pi}$, and the arguments of the two lower marked points, taking as origin the point just below the upper marked point, are $-\frac{\theta}{2}$ and $-\frac{\theta+1}{2}$.

The *dyadic case* is more general. Again, the quotient tree is a dyadic tree. Each cylinder has 3 marked points: one critical point of G on its upper boundary (except for the overcritical cylinder), and two on its lower boundary, to be collapsed to one point. The cylinders are labeled by words in the alphabet $\{0, 1\}$; each cylinder C_w has two "children" $C_{w,0}$ and $C_{w,1}$.

One can make a model for a Cantor set of dyadic type (or rather for its complement). Take the dyadic tree with nodes corresponding to the elements of the set S of words in $\{0, 1\}$ (that carries no information). Assign to each node s a height $G(s)$ and two arguments $\theta'(s)$, $\theta''(s)$. We require that $G(s') < G(s)$ if s' is after s, and that $G(s)$ tends to 0 along an infinite branch of the tree. We then associate to each segment of the tree a cylinder. To the root segment, a cylinder with one-sided infinite height and circumference 2π; on the lower boundary we mark two points with the assigned arguments (and we take record of the generatrix of argument 0). To any other segment E_w a cylinder C_w with one marked point on the upper boundary and two on the lower boundary. The height of C_w is the difference between the heights of the nodes corresponding to its boundary, the arguments of the lower marked points taking the point

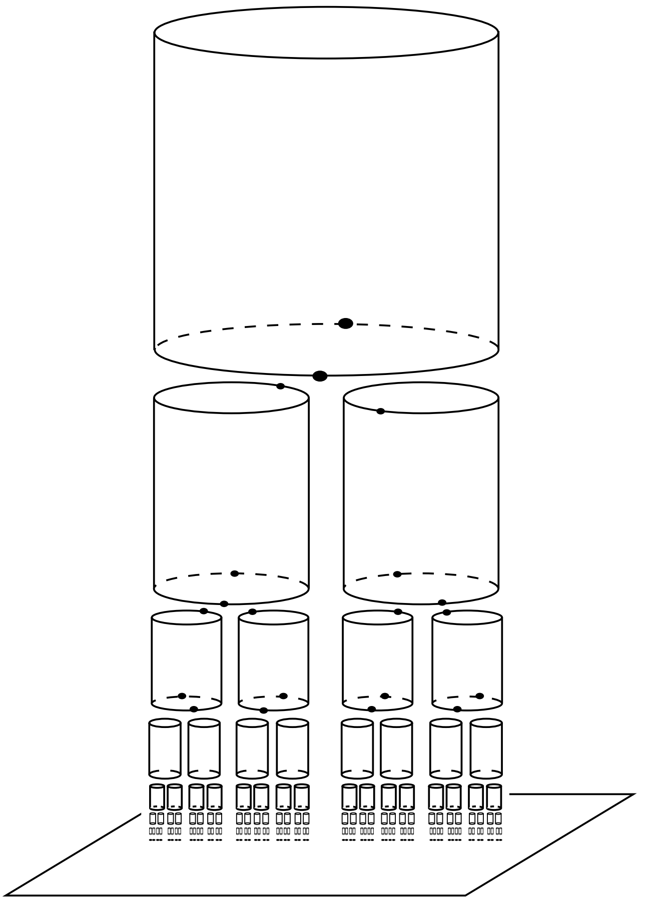


Figure 2: The Euclidean harmonic metric

just below the upper one as origin are the assigned arguments. The two lower marked points are pinched to one point, transforming the lower boundary into a figure eight, of which each component is sewn isometrically to the upper boundary of one of the children cylinders attaching marked point to marked point. This determines the circumferences of all cylinders, and also the moduli.

This way, according to a model given by the data $(G(s), \theta'(s), \theta''(s))_{s \in S}$ we can construct a Riemann surface X. We say that a model is *quasi-quadratic* if the moduli of the cylinders are bounded away from 0 and above, and the ratio of circumference of children cylinders to their parent bounded away from 0 and 1, or equivalently the angle $|\theta'(s) - \theta''(s)|$ is uniformly bounded away from 0 and 1.

Proposition 1. *The Riemann surface given by a quasi-quadratic model can be imbedded in C as the complement of a Cantor set in a unique way, up to postcomposition with an affine map.*

For a proof, see [AB]. Ahlfors and Beurling prove a stronger theorem: it suffices that the sum of the moduli of the intercritical annuli surrounding any point of K is infinite.

3. The Branner-Hubbard motion. Consider the right half-plane $\mathbf{H}_+ = \{\lambda \in \mathbf{C} \mid Re(\lambda) > 0\}$. For $\lambda \in \mathbf{H}_+$, let $\phi_\lambda : \mathbf{C} \to \mathbf{C}$ be the $\mathbf{R}$-affine map which induces the identity on $i\mathbf{R}$ and maps 1 to λ, i.e., $\phi_\lambda(z) = \frac{1}{2}((\lambda+1)z + (\lambda-1)\bar{z})$; let μ_λ denote the Beltrami form $\frac{\bar\partial}{\partial}(\phi_\lambda)$, i.e., $\mu_\lambda = \frac{\lambda-1}{\lambda+1} d\bar{z}/dz$. We denote by μ_λ^K the Beltrami form on $\mathbf{C} - K$ whose expression in the harmonic charts is μ_λ (undefined at critical points of G). Extend μ_λ^K to $\mathbf{C}$ by taking it equal to 0 on K. By the integrability theorem, there is a quasi-conformal mapping $\varphi_\lambda^K : \mathbf{C} \to \mathbf{C}$ such that $\frac{\bar\partial}{\partial}\varphi_\lambda^K = \mu_\lambda^K$. We normalize φ_λ^K in the following way: For the closed unit disc $\overline{\mathbf{D}}$, define $\varphi_\lambda^{\overline{D}}$ by $\varphi_\lambda^{\overline{D}} \circ \exp = \exp \circ \phi_\lambda$. For K arbitrary, we have $\mu_\lambda^K = (\beta^K)^* \mu_\lambda^{\overline{\mathbf{D}}}$ with β^K defined above, so for any choice of φ_λ^K the map $\eta = \eta_\lambda^K = \varphi_\lambda^K \circ (\beta^K)^{-1} \circ (\varphi_\lambda^{\overline{\mathbf{D}}})^{-1}$ is holomorphic at the neighbourhood of ∞; we chose φ_λ^K so that $\eta(z) = z + o(1)$.

Proposition 2. *For each* $z \in \mathbf{C}$*, the point* $\varphi_\lambda^K(z)$ *depends holomorphically on* λ.

Proof. The map $\varphi_\lambda^{\overline{\mathbf{D}}}$ depends holomorphically on λ, and so does the Beltrami form μ_λ^K. Define $\tilde\varphi_\lambda^K$ by $\frac{\bar\partial}{\partial}\tilde\varphi_\lambda^K = \mu_\lambda^K$, $\tilde\varphi_\lambda^K(0) = 0$, $\tilde\varphi_\lambda^K(1) = 1$. By Ahlfors-Bers, $\lambda \mapsto \tilde\varphi_\lambda^K(z)$ is holomorphic for each z. The map β^K does not depend on λ. So the map $\tilde\eta_\lambda^K = \tilde\varphi_\lambda^K \circ (\beta^K)^{-1} \circ (\varphi_\lambda^{\overline{\mathbf{C}}})^{-1}$ is a holomorphic map whose graph undergoes a holomorphic motion parametrized by $\lambda \in \mathbf{H}_+$. It is known ([PT]) that this implies that $(\lambda, z) \mapsto \tilde\eta_\lambda^K(z)$ is holomorphic where defined (say for λ close to some λ_0 and z close to ∞). One can write $\tilde\eta(z) = a(\lambda).z + b(\lambda) + o(1)$ when $z \to \infty$. Then $a(\lambda)$ and $b(\lambda)$ depend holomorphically on λ, and so does $\varphi_\lambda^K(z) = \frac{1}{a(\lambda)}(\tilde\varphi_\lambda^K(z) - b(\lambda))$. □

We set $K_\lambda = \varphi_\lambda^K(K)$. As we have supposed K monic, K_λ is monic for each λ, for the harmonic measure we have $\omega_{K_\lambda} = (\varphi_\lambda^K.)_*(\omega_K)$, and for the separation level $G^{sep}(K_\lambda) = Re(\lambda) \cdot G^{sep}(K)$. It follows that, if (λ_n) is a sequence of parameters such that $Re(\lambda_n) \to 0$ and K_{λ_n} has a limit L, then L is connected.

When we restrict λ to take values in $]0, 1]$, we speak of Branner-Hubbard *compression*. If we restrict it to values in $1 + i\mathbf{R}$, we speak of Branner-Hubbard *turning*.

4. Dendrites

Proposition 3 and Definition. *Let X be a metrizable compact space, connected and locally connected* (*thus arcwise connected*).

The following conditions are equivalent:

(i) *X is homeomorphic to a compact set in $\mathbf{R}^2$, full with empty interior;*
(ii) *X is uniquely arcwise connected;*
(iii) *X is a projective limit of finite trees, with retractions.*

If these conditions are satisfied, we say that X is a **dendrite**.

Comments. The condition "K is full" means that $\mathbf{R}^2 - K$ is connected.

Condition (ii) means that, for any two distinct points a and b in X, there is a unique topological arc $[a, b]_X$ in X with endpoints a and b.

Condition (iii) means that there is a sequence (A_n) of finite topological trees, with inclusions $i_n : A_n \to A_{n+1}$ and continuous retractions $\rho_n : A_{n+1} \to A_n$, and X is homeomorphic to $\varprojlim (A_n, \rho_n)$. Note that this condition implies that X is locally connected.

Proof (i) $\Rightarrow$ (ii)**.** Let I and J be two arcs from a to b, suppose $I \not\subset J$ and let $]c, d[_I$ be a connected component of $I \setminus J$. Then $\Gamma = [c, d]_I \cup [c, d]_J$ is a Jordan curve contained in X, it bounds a disc Δ which is contained in W since X is full. This contradicts the fact that X has empty interior (cf [D/M])

(ii) $\Rightarrow$ (iii): Since X is compact and locally connected, it is uniformly locally connected, i.e., there is a function $h : \mathbf{R}_+ \to \mathbf{R}_+$, continuous at 0 with $h(0) = 0$, such that $\operatorname{diam}([x, y]_X \le h(d(x, y))$. If $A \subset X$ is a closed connected subset, define a map $p_A : X \to A$ in the following way: choose a point $a_0 \in A$; for any point $x \in X$, the set $[a_0, x]_X \cap A$ is of the form $[a_0, y]_X$, then set $p_A(x) = y$. The map p_a does not depend on the choice of a_0, and it is a retraction. If $p_A(x) \neq p_A(y)$, one has $[x, y]_X = [x, p_A(x)]_X \cup [p_A(x), p_A(y)]_X \cup [p_A(y), y]_X$, therefore $d(p_A(x), p_A(y)) \le h(d(x, y))$ and p_A is continuous.

Let $\{a_n\}_{n \in \mathbf{N}}$ be a dense set in X. For each n, the set $A_n = [a_0, \dots, a_n]_X = \bigcup_{i \le n} [a_0, a_i]_X$ is a finite tree in X. Set $p_n = p_{A_n} : X \to A_n$ and denote by ρ_n the restriction of p_n to A_{n+1}. Then $x \mapsto (p_n(x))_n$ is a homeomorphism of X onto $\varprojlim (A_n, \rho_n)$.

(iii) $\Rightarrow$ (i): Suppose $X = \varprojlim (A_n, \rho_n)$. Choose a sequence ϵ_n of positive numbers such that $\sum \epsilon_n < \infty$. By induction, construct an imbedding $\iota_n : A_n \to \mathbf{R}^2$ so that $\iota_{n+1}|_{A_n} = \iota_n$ and $(\forall x \in A_{n+1})\ d(\iota_{n+1}(x), \iota_n(\rho_n(x)) \le \epsilon_n$. Then, for any $x = (x_n) \in X$, the sequence $(\iota_n(x))$ converges to a point $\iota(x) \in \mathbf{R}^2$, and the map ι defined this way is a continuous embedding of X in $\mathbf{R}^2$. The space X has topological dimension 1, therefore $\iota(X)$ has empty interior in **R2**. $\square$

Proposition 4. *Let X be a dendrite.*

a) *Any closed connected subset Y of X is a dendrite.*
b) *Let $\mathcal{R}$ be an equivalence relation on X. Suppose that*

- *the graph of $\mathcal{R}$ is closed in $X \times X$;*

- *each equivalence class is connected.*

Then $Z = X/\mathcal{R}$ *is a dendrite.*

Proof. a) Write $X = \underset{\leftarrow}{\lim}(A_n, \rho_n)$ and denote by B_n the image of Y in A_n. Then, for each n, the set B_n is a finite topological tree, ρ_n induces a continuous retraction $B_{n+1} \to B_n$ and $Y = \underset{\leftarrow}{\lim} B_n$.

b) Since $\mathcal{R}$ is closed, $Z = X/\mathcal{R}$ is Hausdorff. Denote by p the projection $X \to Z$. Since X is compact, connected, locally connected and arcwise connected, the same holds for Z; let us show that Z is uniquely arcwise connected. Let a and b be two points in Z, $a' \in p^{-1}(a)$, $b' \in p^{-1}(b)$ and γ an arc from a to b. Since the fibers of p are connected, the compact set $\Gamma = p^{-1}(\gamma)$ is connected, so Γ contains the unique arc $\gamma' = [a', b']_X$ and $p(\gamma') \subset \gamma$. But the set $p(\gamma')$ is connected and contains $\{a, b\}$, so $p(\gamma') = \gamma$. Now the arc $\gamma' = [a', b']_X$ does not depend on the choice of γ, therefore $\gamma = p(\gamma')$ is unique. □

5. Bridging Cantor sets. Let K be a Dirichlet regular compact set in $\mathbf{C}$ (we are thinking of a Cantor set), denote by G the potential. An *external ray* is a line in $\mathbf{C} - K$ orthogonal to the equipotentials; when oriented towards decreasing potential, we call it a *descending ray*. A descending ray may tend to K or bump on a critical point of G. A *critical link* is an arc of external ray joining a critical point of G to another critical point of G. A *descending line* is a piecewise $\mathbf{R}$-analytic topological arc (compact or not) which is a union of arcs of external rays and critical links, oriented towards decreasing potential. When continued, a descending line tends to K, it may land at a point of K or wander in a connected component. If K is a Cantor set, any descending line lands at a point of K.

For z and z' two points in $\mathbf{C} - K$, we say that z' is *below* z if there is a descending line from z to z'. For a a critical point of G in $\mathbf{C} - K$, the *bridge* of a is the closure in $\mathbf{C}$ of the set of points in $\mathbf{C} - K$ which are below a (including a itself); the bridge of a is compact, connected and contains at least two points of K.

We denote by $K^{bridged}$ the union of K and the bridges of all critical points of G.

Proposition 5. *If* K *is a Cantor set,* $K^{bridged}$ *is a dendrite.*

Proof. For $\epsilon > 0$, let L_ϵ be the union of $G^{-1}([0, \epsilon])$ and the bridges of critical points of level $> \epsilon$, and X_ϵ the quotient of L_ϵ by the equivalence relation contracting the region enclosed by each equipotential component of level ϵ to a point. Then, each X_ϵ is topologically a finite tree, and $K^{bridged} = \underset{\leftarrow}{\lim}\ X_\epsilon$ is a dendrite. □

On $K^{bridged}$ we consider the equivalence relation $\sim_B^*$ algebraically generated by collapsing each bridge: x and y are equivalent if there is a finite sequence $(x_0 = x, \ldots, x_n = y)$ such that x_{i-1} and x_i are in the bridge of a common critical point of G for $i = 1, \ldots, n$.

We define the equivalence $\sim_B$ by its graph, which is the closure of that of $\sim_B^*$, and we denote by $K^\bullet$ the quotient space $K^{bridged}/\sim_B$.

We say that there is a *landing coincidence* at a point $x \in K$ if there are two descending lines from different critical points of G, not contained in one another, both landing at x. If there is no landing coincidence, $x \sim_B^* y$ means that $x = y$ or x and y belong to a common bridge; then the graph of $\sim_B^*$ is closed, and $\sim_B$ coincides with $\sim_B^*$.

Proposition 6. *If K is a Cantor set, $K^\bullet$ is a dendrite* (*possibly reduced to a point*).

Proof. In view of Proposition 4, (b), it suffices to prove that the classes of $\sim_B$ are connected. If $x \sim_B y$, one can find sequences (x_n) tending to x and (y_n) tending to y such that $x_n \sim_B^* y_n$. For each n, one can find a connected compact set Λ_n (a finite union of bridges) containing x and y, and contained in their $\sim_B^*$ class. From the sequence $(\Lambda_n)_n$, one can extract a subsequence converging for the Hausdorff metric to a connected compact set Λ, containing x and y and contained in their $\sim_B$ class. □

Definition. Let K be a Dirichlet regular Cantor set in $\mathbf{C}$. I say that K is **dendrogenous** if, under Branner-Hubbard compression, K_λ has a limit L when $\lambda \in]0, 1]$ tends to 0, and L is homeomorphic to $K^\bullet$.

6. Examples from quadratic Julia sets. We set $f_c(z) = z^2 + c$ and $K_c = K(f_c)$. For $c \in \mathbf{C} - M$, the Branner-Hubbard compression of K_c gives $(K_{c_t})_{t \in [0,1]}$, where c_t is the point on the external ray $R_M(\theta)$ passing through c with $G_M(c_t) = t \cdot G_M(c)$.

For $c \in \mathbf{R}_+ \setminus M =]\frac{1}{4}, \infty[$, the bridge of 0 is an infinite tree whose closure contains K_c, the set $K_c^\bullet$ is reduced to a point. The point c_t tends to $\frac{1}{4}$ along $\mathbf{R}_+$, we are in the situation of parabolic implosion, K_{c_t} does not have a limit but a family parametrized by $\mathbf{T}$ of possible limits, none of which is reduced to a point ([D/AMS], [L]). For such a c, the Cantor set K_c is not dendrogenous.

Let θ be an irrational angle such that the external ray $R_M(\theta)$ lands on a point c_0 on the boundary of a hyperbolic component of M. For $c \in R_M(\theta)$ the set K_{c_t} tends to K_{c_0} ([D/AMS]), but K_{c_0} either has non empty interior or is not locally connected, so K_c is not dendrogenous. Note that, if c_0 is on the main cardioid, α_c is linked to $-\alpha_c$ by the bridge of 0; in fact α is linked by bridges to an infinity of its iterated preimages, and by finite chains of bridges to all its iterated preimages. In that case, $K_c^\bullet$ is reduced to a point; and there are infinitely many infinite landing coincidences.

If $R_M(\theta)$ accumulates on an infinitely renormalizable point $c_0 \in M$, the set K_{c_0} can be locally connected (for instance if $c_0 \in \mathbf{R}$) or not locally connected ([S]). In case $c_t \to c_0$ and K_{c_0} is locally connected, it is a dendrite, K_{c_t} tends to K_{c_0}, but I don't know if K_{c_0} is necessarily homeomorphic to $K_c^\bullet$.

The following theorem gives examples of dendrogenous Cantor Julia sets.

Theorem 1. *Let $\theta \in \mathbf{T}$ be an angle such that the external ray $R_M(\theta)$ lands at a point $c_0 \in M$ such that the map f_{c_0} is not renormalizable and has no indifferent fixed point. Then, for any $c \in R_M(\theta)$, the Cantor set K_c is dendrogeneous.*

Proof. By hypothesis, c_t tends to c_0 when $t \to 0$. By results of Yoccoz ([B/AMS], [H/M], [Y]), the set $K(f_{c_0})$ is a dendrite (the proof involves some facts which are specific of degree 2). The map $f \mapsto K(f)$ is continuous at f_{c_0} ([D/AMS]); so $K_{c_t} \to K_{c_0}$. It remains to show that K_{c_0} is homeomorphic to $K_c^\bullet$. To fix the ideas we shall introduce the Yoccoz puzzles:

Let $\tau_1, \ldots, \tau_q$ be the external arguments of the fixed point α_{c_0} of f_{c_0} (the one which does not have 0 as external argument). For c close to c_0, the rays $R_c(\tau_i)$ land at α_c. Fix a potential s. The puzzle pieces of depth r for f_c with c close to c_0 are the closures of the connected components of $f^{-r}(G_c^{-1}([0, s[)\setminus(\{\alpha_c\} \cup R_c(\tau_1) \cup \ldots \cup R_c(\tau_q)))$. The result of Yoccoz is that the puzzle pieces of various depth containing a point $x \in K_{c_0}$ form a fundamental system of neighbourhoods of x (if x is a preimage of α_0, take for each r the union of pieces of depth r containing x). For each r, there is a neighbourhood V_r of c_0 such that, for $c \in V_r$, the puzzle for f_c has the same combinatorics as the puzzle of f_{c_0} down to depth r.

Let x be a point in K_c and for $t \in]0, 1]$ let x_t be its image under B-H compression. For each level r, for t suficiently small x_t is in a puzzle piece for K_c correponding to a piece P_r of K_{c_0}. These pieces nest down to a point $x_0 \in K_{c_0}$ and $x_t \to x_0$ when $t \to 0$. This way we define a continuous map $\rho : K_c \to K_{c_0}$.

Given x, y, z in $\mathbf{C}$, with x a critical point of G_c or a repelling periodic point of f, we say that x separates y from z (with respect to f) if the union of $\{x\}$ and the rays descending to x separates y from z.

If x and y in K_c are linked by a bridge, they are not separated by preimages of α_c, and it follows that $\rho(x) = \rho(y)$. So ρ gives a map $\tilde{\rho} : K_c^\bullet \to K_{c_0}$. It is clear that this map is continuous and surjective, let us show that it is injective (thus a homeomorphism).

Lemma 1. *Under the hypotheses of the Theorem, let c be a point of $R_M(\theta)$ and ω', ω'' two critical points of G_c of different level (i.e., $G_c(\omega') \neq G_c(\omega'')$). Then there is an iterated preimage of α_c separating ω' from ω''.*

Proof. The points ω' and ω'' have external arguments θ', θ'' (with respect to K_c), with $2^{k'}\theta' = 2^{k''}\theta'' = \theta$ and $k' \neq k''$. Then $R_{c_0}(\theta')$ and $R_{c_0}(\theta'')$ land at points ω'_0 and ω''_0 such that $f_{c_0}^{k'}(\omega') = f_{c_0}^{k''}(\omega'') = c_0$. The point c_0 is not periodic since K_{c_0} has empty interior, so $\omega'_0 \neq \omega''_0$. By the result of Yoccoz, ω'_0 and ω''_0 are separated by an iterated preimage α'_0 of α_{c_0}. There is a point α' whose external arguments with respect to f_c are the same as those of α'_0 with respect to f_{c_0}, and this point α' separates ω' from ω''. ☐

End of the proof of Theorem 1: Let x and y be two distinct points of K_c such that $\rho(x) = \rho(y)$. This means that x and y are not separated by iterated preimages of α_c. Let ω be the highest critical point of G_c separating x from y. The bridge of ω lands at points x' and y' in K_c, on the side of x and y respectively. Suppose $x \neq x'$. Then there is a highest critical point ω' separating x from x', and $G_c(\omega') < G_c(\omega)$. By the lemma, there is an iterated preimage α' of α_c separating ω from ω', and this point α' separates ω and y from x. This is a contradiction, therefore $x = x'$. Similarly $y = y'$, so x and y are linked by a bridge, they have the same image in $K_c^\bullet$.

This proves that the map $\tilde{\rho}$ is injective. □

Corollary. *For almost all c on an equipotential in $\mathbf{C} - M$, the Cantor set K_c is dendrogenous.*

Proof. The values of θ such that $R_M(\theta)$ accumulates on a point c such that f_c is renormalizable or has an indifferent fixed point are given by algorithms described in [D/A]. They form a set of measure 0. For the other values of θ, the ray $R_M(\theta)$ lands at a point of M (Yoccoz), and we are in the conditions of the theorem. □

7. Two conjectures. The case of quadratic Cantor Julia sets suggests the following conjecture:

Conjecture 1. *Let K be a quasi-quadratic Cantor set, and let (K_θ) be the family obtained from K by Branner-Hubbard turning. Then, for almost all θ, the Cantor set K_θ is dendrogenous.*

The Branner-Hubbard turning consists, for each cylinder C_i, to turn the lower boundary with its marked points by an angle $-b \cdot h_i$, where h_i is the height of C_i.

More generally, we can turn the lower boundary of C_i by an angle $-\sigma_i$, these angles being chosen independently. We remain in the quasi-quadratic case. This way we obtain a family (K_σ) with $\sigma = (\sigma_i)$ ranging in $\mathbf{T}^I$, where $I = \mathrm{Crit}(G_K)$ is the set of nodes of the quotient tree.

The space $\mathbf{T}^I$ carries a natural uniformly spread measure, so we can formulate:

Conjecture 2. *In this situation, the Cantor set K_σ is dendrogenous for almost all $\sigma \in \mathbf{T}^I$.*

8. The semi-hyperbolic case. A point $c \in \partial M$ is called *semi-hyperbolic* if f_c has no parabolic cycle and the critical point 0 is not recurrent under f_c (note that, for $c \in \partial M$, attracting cycles are excluded). A dendrite $K \in \mathbf{C}$ is called a *John dendrite* if

$$(\exists \lambda > 0)(\forall \theta \in T)(\forall z \in \mathcal{R}_K(\theta)) \qquad d(z, K) \leq \lambda |z - \gamma_K(\theta)|.$$

Here $\mathcal{R}_K(\theta)$ denotes the external ray of K of argument θ and $\gamma_K(\theta)$ its landing point. L. Carleson, P. Jones and J.-C. Yoccoz have proved ([CJY]) that, for $c \in \partial M$, the filled Julia set K_c is a John dendrite if and only if c is semi-hyperbolic.

Theorem 2. *For $\theta \in \mathbf{T}$, the following conditions are equivalent:*

(1) *θ is non-recurrent under doubling;*
(2) *$\mathcal{R}_M(\theta)$ lands at a semi-hyperbolic point $c_0 \in \partial M$;*

(3) *For $c \in \mathcal{R}_M(\theta)$, the Cantor set $K = K_c$ has no critical link and satisfies the following properties:*

(3.1) *There is an $\alpha > 0$ such that $\forall i$ the arguments in C_i of two rays descending from higher critical points of G_K differ by at least α;*

(3.2) *There is a $\mu \in \mathbf{N}$ such that two rays descending from critical points of G_K cross together at most μ consecutive cylinders.*

Lemma 2. *Let $c_0 \in \partial M$ be an accumulation point of an external ray $\mathcal{R}_M(\theta)$.*

a) *If c_0 is of the form $c' \perp c''$ (c' tuned by c''), where $f_{c'}$ has a superattracting cycle of order k and c'' does not admit a dyadic argument, then $(\exists n > 1)\ |2^n\theta - \theta| \leq 2^{-k}$.*

b) *If c_0 is infinitely renormalizable, then θ is recurrent under doubling.*

c) *If f_{c_0} has an indifferent cycle, then θ is recurrent under doubling.*

Proof. a) We use the tuning algorithm for external arguments ([D/A]): Let θ' and θ'' be the root arguments of c', and t the (by hypothesis non dyadic) argument of a ray accumulating on c'' corresponding to θ, then, knowing the expansions in base 2 $\theta' = \cdot\overline{u_1 \dots u_k}$ $\quad \theta'' = \cdot\overline{v_1 \dots v_k}$ $\quad t = \cdot s_1 s_2 \dots s_n \dots$, we get for θ:

$$\theta = \cdot w_{1,1} \dots w_{1,k} w_{2,1} \dots w_{2,k} \dots w_{n,1} \dots w_{n,k} \dots$$

where $w_{n,i} = u_i$, if $s_n = 0$ and v_i, if $s_n = 1$. Take $n > 1$ such that $s_n = s_1$ (this is possible because in particular $c'' \neq -2$), then θ and $2^{kn}\theta$ have the same k first digits.

b) It is an immediate consequence of (a), as c_0 can be written as $c' \perp c''$ in infinitely many ways, with c'' not having a dyadic external argument in M.

c) If f_{c_0} has an indifferent fixed point with multiplier $e^{2\pi i s}$, then θ is given by the staircase algorithm ([D/A], [BS]). It follows that any initial sequence of digits is repeated somewhere. Thus θ is recurrent under doubling. If f_{c_0} has an indifferent cycle of order $k > 1$, then the point c_0 is of the form $c' \perp c''$, where $f_{c'}$ has a superattracting cycle of order k and $f_{c''}$ has an indifferent fixed point. Then θ is given by the tuning algorithm. With the notations of (a), in the expansion of t any initial sequence of digits is repeated, so in the expansion of θ any initial sequence of length nk is repeated somewhere. □

Proof of Theorem 2. a) (1) $\Longrightarrow$ (2): Suppose θ is non recurrent under doubling and let c_0 be an accumulation point of $\mathcal{R}_M(\theta)$. By Lemma 2, f_{c_0} is not infinitely renormalizable and has no indifferent cycle. Then by Yoccoz ([H/M]), M is locally connected at c_0, the ray $\mathcal{R}_M(\theta)$ does land at c_0, and K_{c_0} is a dendrite. Let us show that 0, or equivalently c_0, is not recurrent under f_{c_0}. If c_0 has only θ as external argument in K_{c_0}, then $f^n_{c_0}(c_0)$ cannot approach c_0.

Suppose that c_0 has two external arguments, θ and another one θ', in $K = K_{c_0}$. A priori, it might happen that θ is not recurrent, but $\theta' \in \omega(\theta)$ (ω limit set under doubling), making c_0 recurrent. Let us show that this is impossible. Let $\tau = \frac{p}{2^k} \in\,]0, 1[$ be the leading dyadic number (i.e., with k minimal) between θ and θ'. Let z_0 be the landing point of $\mathcal{R}_{c_0}(\tau)$, set $z_n = f^n_{c_0}(z_0)$ (so that $z_k = \beta$, the landing point of $\mathcal{R}_{c_0}(0)$) and $z_{k-1} = -\beta$, the landing point of $\mathcal{R}_{c_0}(1/2)$, and let $A = [z_0, \dots, z_k]_K$ be the connected envelope of

$\{z_0, \ldots, z_k\}$ in K. Then A is a finite tree, $0 \in A$, because $0 \in [-\beta, \beta]_K$ and $c_0 \in A$ because the ray $\mathcal{R}_{c_0}(\tau)$ landing at z_0 is separated from z_k by $\mathcal{R}_{c_0}(\theta) \cup \mathcal{R}_{c_0}(\theta') \cup \{c_0\}$. It follows that $f_{c_0}(A) \subseteq A$ and hence the orbit of 0 is in $f_{c_0}(A) \subset A$, since $f_{c_0}(\{0, z_0, \ldots, z_k\}) = \{c_0, z_1, \ldots, z_k\}$. For each n, the rays $\mathcal{R}_{c_0}(2^n\theta)$ and $\mathcal{R}_{c_0}(2^n\theta')$ land at the same point $x_n = f_{c_0}^{n+1}(0)$. Moreover they are on opposite sides of the branch of A passing through x_n, because they have different arguments (they could only have the same argument at some time n, if they previously passed through the critical point 0, which would make 0 periodic). Therefore, if $2^{n_j}\theta$ tends to θ', then x_{n_j} tends to c_0 and $2^{n_j}\theta'$ tends to θ, so $\theta \in \omega(\theta')$ and finally $\theta \in \omega(\theta)$, countrary to the hypothesis.

If c_0 has at least three external arguments $\theta_1, \theta_2, \theta_3$ with $0 < \theta_1 < \theta_2 < \theta_3 < 1$, we can consider the leading dyadic numbers τ_1 in $]\theta_1, \theta_2[$ and τ_2 in $]\theta_2, \theta_3[$, the landing points z_1 of $\mathcal{R}_{c_0}(\tau_1)$ and z_2 of $\mathcal{R}_{c_0}(\tau_2)$, and the connected envelope A of the union of the orbits of z_1 and z_2. Then as above A is a finite tree, $0, c_0 \in A$, $f_{c_0}(A) \subset A$, the point c_0 is a branching point of A, and so is $f_{c_0}^n(c_0)$ for each n. As the number of branch points is finite the orbit of c_0 is finite and c_0 is a Misiurewicz point.

b) (2) $\Longrightarrow$ (1): If c_0 is semi-hyperbolic, K_{c_0} is locally connected ([CJY]). Then the Caratheodory loop $\gamma : \mathbf{T} \to K_{c_0}$ is continuous. If θ were recurrent under doubling, then c_0 were recurrent under f_{c_0}.

c) (1) $\Longrightarrow$ (3.1): Note first that critical links occur only for θ periodic under doubling, i.e., rational with odd denominator. Suppose θ is non recurrent under doubling and set $\alpha = d(\theta, \omega(\theta))$. Let C be a cylinder with $G(\partial^- C) = \frac{G(0)}{2^k}$, and let $\mathcal{R}'$ and $\mathcal{R}''$ be two consecutive descending rays in C, coming from precritical points x' and x'' with $f_c^{k'}(x') = f_c^{k''}(x'') = c$. We have $k' \neq k''$, because an equipotential component contains at most one critical point of G. Suppose $k'' < k'$. Let C_0, C' and C'' be the cylinders having c, x' and x'' in their upper boundary. In C_0, the rays $f_c^{k'}(\mathcal{R}')$ and $f_c^{k'}(\mathcal{R}'')$ have argument θ and $\tilde{\theta} = 2^{k'-k''}\theta$ respectively, these arguments differ by at leats α. The map $f_c^{k'}$ induces an isomorphism $C' \to C_0$ preserving argument differences, so in C' the rays $\mathcal{R}'$ and $\mathcal{R}''$ have arguments which differ by at least α. The strip between $\mathcal{R}'$ and $\mathcal{R}''$ goes down to C without hitting a precritical point, because $\mathcal{R}'$ and $\mathcal{R}''$ are consecutive. So the metric width of this strip remains the same, and its argument width is multiplied by $2^{k-k'}$. We have seen that the difference of arguments between two descending rays is at least α along an arc on which they are consecutive. Naturally this holds also without this consecutiveness hypothesis.

d) (2) $\Longrightarrow$ (3.2): Suppose $\mathcal{R}_M(\theta)$ lands at $c_0 \in \partial M$ semi-hyperbolic. By (2) $\Longrightarrow$ (1) and Lemma 2, f_{c_0} is only finitely renormalizable and has no indifferent cycle. By [H/M], the pieces of Yoccoz puzzles containing c_0 form a fundamental system of neighbourhoods of c_O. So, in K_{c_0} there is a finite number of repelling preperiodic points $z_i(c_0)$ which, together with the external rays $\mathcal{R}_{c_0}(t_{i,j})$ landing on them, separate c_0 and $\mathcal{R}_{c_0}(\theta)$ from $f_{c_0}^n(c_0)$ and $\mathcal{R}_{c_0}(2^n\theta)$ for $n > 0$. For each $n > 0$, if c is close enough to c_0, the corresponding preperiodic points $z_i(c)$ together with the rays $\mathcal{R}_c(t_{i,j})$ separates $\mathcal{R}_c(\theta)$ from $\mathcal{R}_c(2^n\theta)$. Actually this holds for any $c \in \mathcal{R}_M(\theta)$, since the combinatorics of external rays does not change when c moves on an external ray of M. Fix $c \in \mathcal{R}_M(\theta)$, let s_i be the separation potential between c and $z_i(c)$, choose $s < \min(s_i)$ a non critical level, let E be the equipotential component of level s of K_c, which intersects $\mathcal{R}_c(\theta)$ and C_1 the cylinder which contains E. Then no ray of the form $\mathcal{R}_c(2^n\theta)$ with $n > 0$ crosses C_1. There is a $\mu \in \mathbf{N}$ such that $f_c^{\mu}(C_1) = C_0$ (the cylinder with c on its upper boundary). Let $\mathcal{R}$ ans $\mathcal{R}'$ be two distinct rays descending from precritical points x and x' with $f^k(x) = f^{k'}(x') = 0$, $k' \le k$. Let C be the cylinder with x on its upper boundary, so that $\mathcal{R}$ crosses C. Then $f_c^{k+1}(C) = C_0$, and f_c^{k+1} induces an isomorphism of C together with everything below to C_0 together with everything below. Suppose that $\mathcal{R}'$ also crosses C, which implies $x' \neq x$ and $k' < k$. In C_0, $f_c^{k+1}(\mathcal{R}) = \mathcal{R}_c(\theta)$ and $f_c^{k+1}(\mathcal{R}') = \mathcal{R}_c(2^{k-k'}\theta)$, since they are rays descending from c and $f_c^{k-k'}(c)$ respectively. These two rays can be continued downwards without hitting a critical point. (there is no critical link, so no ray descending from a postcritical point of f_c can hit a precritical point). If $\mathcal{R}$ and $\mathcal{R}'$ stay in the same cylinder μ' times, then so do $\mathcal{R}_c(\theta)$ and $\mathcal{R}_c(2^{k-k'}\theta)$. Therefore $\mu' \le \mu$, since $\mathcal{R}(\theta)$ crosses C_1 and $\mathcal{R}(2^{k-k'}\theta)$ cannot enter it.

e) (No critical link) and (3.2) $\Longrightarrow$ (1): The angle θ is not periodic under doubling (i.e., not rational with odd denominator) because there is no critical link. Suppose, by contradiction, that θ is recurrent under doubling and choose θ' with $2\theta' = \theta$, so that $\mathcal{R}_c(\theta')$ bumps on the critical point 0. For any k, there is an n such that $2^n\theta = \theta + \epsilon$ is so close to θ that there is no t with $2^l t = \theta$, $l \le k$, between θ and $2^n\theta$. Set $\theta'' = \theta' - \frac{\epsilon}{2^{n+1}}$. Then $2^{n+1}\theta'' = 2^n\theta - \epsilon = \theta$, so $\mathcal{R}_c(\theta'')$ bumps on a precritical point ω' with $f_c^n(\omega') = 0$.

Suppose that a ray $\mathcal{R}_c(t')$ with t' (strictly) between θ' and θ'' bumps on a precritical point w with $f_c^m(w) = 0$. Then $t = 2^n t'$ lies between θ and $2^n\theta$. We cannot have $m \le n$ because $2^{m+1}t' = \theta$ and we would have $t = 2^{n-m}\theta$. Neither can we have $n \le m \le n + K$ because $2^l t = \theta$ with $l = n - m$. So $m > n + k$.

Therefore the rays $\mathcal{R}_c(t')$ with t' between θ' and θ'' span an open strip which crosses $n + k + 1$ consecutive cylinders without hitting a critical point of the potential (i.e., a precritical point of f_c). Below the level of ω', this strip is bounded by a ray descending from ω', and by another ray descending from 0, and these rays stay together across k consecutive cylinders.

As k is arbitrary, this contradicts (3.2). □

Remarks. 1) We have proved more than stated: the statement says $\{$(3.1) and (3.2)$\}$ $\Longrightarrow$ (1), and we have proved (3.2) $\Longrightarrow$ (1). But we have in view the conjecture below. In the quadratic case, the conditions (3.1) and (3.2) are more or less equivalent; it is not clear that the same holds for general quasi-quadratic Cantor sets, and we shall be happy if we prove this conjecture as stated.

2) We use Lemma 2 (c) only to get local connectivity, which is convenient in the proof of (1) $\Longrightarrow$ (2). Then we just want to exclude parabolic cycles, which is much easier. It might be possible to improve this proof by avoiding this contortion.

Conjecture 3. *If a quasi-quadratic Cantor set satisfies* (3.1) *and* (3.2)*, it is dendrogenous, and under Branner-Hubbard compression it tends to a John dendrite.*

Note that none of the above conjectures implies obviously any other.

References

[AB] L. AHLFORS and A. BEURLING, *Conformal Invariants and Function Theoretic Null Sets*, Acta Math., 1950.

[BMS] I. BINDER, N. MAKAROV and S. SMIRNOV, *Harmonic measures and polynomial Julia sets*, Duke Math. J., 117 (2003) no. 2, pp. 343–365.

[B/AMS] B. BRANNER, *Puzzles and Para-Puzles of Quadratic and Cubic Polynomials*, in *Complex Dynamical Systems, The Mathematics Behind the Mandelbrot and Julia Sets*, Proceedings of Symposia in Applied Mathematics 49 (AMS) pp. 31–70, R. Devaney, editor, Cincinati 1994.

[BH1] B. BRANNER and J. H. HUBBARD, *The iteration of cubic polynomials, part I : The global topology of parameter space,* Acta Math 160 (1988), 14–206.

[BH2] B. BRANNER and J. H. HUBBARD, *The iteration of cubic polynomials, part II : Patterns and parapatterns*, Acta Math, 169 (1992) pp. 229–325.

[CJY] L. CARLESON, P. JONES and J.-C. YOCCOZ : *Julia and John*, Bol. Soc. Bras. Mat. (1994), pp. 1–30.

[D/AMS] A. DOUADY, *Does a Julia Set Depend Continuously on the Polynomial? Complex Dynamical Systems, The Mathematics Behind the Mandelbrot and Julia Sets*, Proceedings of Symposia in Applied Mathematics 49 (AMS), pp. 91–138, R. Devaney, editor, Cincinati 1994.

[D/A] A. DOUADY, *Algorithms for Computing Angles in th Mandelbrot Set*, in *Chaotic Dynamics and Fractals*, edited by M. Barnsley and S. Demko, Atlanta 1985.

[D/M] A. DOUADY, *Description of compact sets in* **C**, in *Topological Methods in Modern Mathematics*, in honour of John Milnor (Publish or Perish), 1993 pp. 429–465.

[D/V] A. DOUADY, *Geometrîay Dinâmica en el complemento de un compacto en* C, proceedings of a meeting in Cuernavaca (2003) in honour of Alberto Verjovsky, to appear in contemporary Mathematics, Published by AMS.

[DH] A. DOUADY and J. H. HUBBARD, *Etude dynamique des polynomes complexes*, I and II, Publications Mathematiques d'Orsay.

[H/M] J. H. HUBBARD, *Local connectivity of Julia sets and bifurcation loci, Three theorems of Yoccoz*, in Topological Methods in Modern Mathematics, in honour of John Milnor (Publish or Perish), 1993.

[L] P. LAVAURS, Thèse, Orsay.

[PT] C. PETERSEN and T. LEI: *Branner-Hubbard motions and attracting dynamics*, in this volume.

[S] D. SØRENSEN, Thesis, DTU.

48 rue Monsieur le Prince
F 75006 Paris, France
e-mail: adrien.douady@ens.fr